جامعة
بنغازي الحديثة

مجلة جامعة بنغازي الحديثة للعلوم والدراسات الإنسانية

مجلة علمية الكترونية محكمة

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شروط كتابة البحث العلمي في مجلة جامعة بنغازي الحديثة للعلوم والدراسات الإنسانية

1- الملخص باللغة العربية وباللغة الانجليزية (150 كلمة).
2- المقدمة، وتشمل التالي:
- ❖ نبذة عن موضوع الدراسة (مدخل).
- ❖ مشكلة الدراسة.
- ❖ أهمية الدراسة.
- ❖ أهداف الدراسة.
- ❖ المنهج العلمي المتبع في الدراسة.

3- الخاتمة. (أهم نتائج البحث - التوصيات).
4- قائمة المصادر والمراجع.
5- عدد صفحات البحث لا تزيد عن (25) صفحة متضمنة الملاحق وقائمة المصادر والمراجع.

القواعد العامة لقبول النشر

1. تقبل المجلة نشر البحوث باللغتين العربية والانجليزية؛ والتي تتوافر فيها الشروط الآتية:

- أن يكون البحث أصيلاً، وتتوافر فيه شروط البحث العلمي المعتمد على الأصول العلمية والمنهجية المتعارف عليها من حيث الإحاطة والاستقصاء والإضافة المعرفية (النتائج) والمنهجية والتوثيق وسلامة اللغة ودقة التعبير.
- ألا يكون البحث قد سبق نشرة أو قُدم للنشر في أي جهة أخرى أو مستل من رسالة أو اطروحة علمية.
- أن يكون البحث مراعياً لقواعد الضبط ودقة الرسوم والأشكال - إن وجدت - ومطبوعاً على ملف وورد، حجم الخط (14) وبخط (Arial 'Body') للغة العربية. وحجم الخط (12) بخط (Times New Roman) للغة الإنجليزية.
- أن تكون الجداول والأشكال مدرجة في أماكنها الصحيحة، وأن تشمل العناوين والبيانات الإيضاحية.
- أن يكون البحث ملتزما بدقة التوثيق حسب دليل جمعية علم النفس الأمريكية (APA) وتثبيت هوامش البحث في نفس الصفحة والمصادر والمراجع في نهاية البحث على النحو الآتي:
- أن تُثبت المراجع بذكر اسم المؤلف، ثم يوضع تاريخ نشرة بين حاصرتين، ويلي ذلك عنوان المصدر، متبوعاً باسم المحقق أو المترجم، ودار النشر، ومكان النشر، ورقم الجزء، ورقم الصفحة.
- عند استخدام الدوريات (المجلات، المؤتمرات العلمية، الندوات) بوصفها مراجع للبحث: يُذكر اسم صاحب المقالة كاملاً، ثم تاريخ النشر بين حاصرتين، ثم عنوان المقالة، ثم ذكر اسم المجلة، ثم رقم المجلد، ثم رقم العدد، ودار النشر، ومكان النشر، ورقم الصفحة.

2. يقدم الباحث ملخص باللغتين العربية والانجليزية في حدود (150 كلمة) بحيث يتضمن مشكلة الدراسة، والهدف الرئيسي للدراسة، ومنهجية الدراسة، ونتائج الدراسة. ووضع الكلمات الرئيسية في نهاية الملخص (خمس كلمات).

3. تحتفظ مجلة جامعة بنغازي الحديثة بحقها في أسلوب إخراج البحث النهائي عند النشر.

إجراءات النشر

ترسل جميع المواد عبر البريد الالكتروني الخاص بالمجلة جامعة بنغازي الحديثة وهو كالتالي:

- ✓ يرسل البحث الكترونياً (Word + Pdf) إلى عنوان المجلة info.jmbush@bmu.edu.ly او نسخة على CD بحيث يظهر في البحث اسم الباحث ولقبة العلمي، ومكان عملة، ومجاله.
- ✓ يرفق مع البحث نموذج تقديم ورقة بحثية للنشر (موجود على موقع المجلة) وكذلك ارفاق موجز للسيرة الذاتية للباحث إلكترونياً.
- ✓ لا يقبل استلام الورقة العلمية الا بشروط وفورمات مجلة جامعة بنغازي الحديثة.
- ✓ في حالة قبول البحث مبدئياً يتم عرضة على مُحكمين من ذوي الاختصاص في مجال البحث، ويتم اختيارهم بسرية تامة، ولا يُعرض عليهم اسم الباحث أو بياناته، وذلك لإبداء آرائهم حول مدى أصالة البحث، وقيمته العلمية، ومدى التزام الباحث بالمنهجية المتعارف عليها، ويطلب من المحكم تحديد مدى صلاحية البحث للنشر في المجلة من عدمها.
- ✓ يُخطر الباحث بقرار صلاحية بحثه للنشر من عدمها خلال شهرين من تاريخ الاستلام للبحث، وبموعد النشر، ورقم العدد الذي سينشر فيه البحث.
- ✓ في حالة ورود ملاحظات من المحكُمين، تُرسل تلك الملاحظات إلى الباحث لإجراء التعديلات اللازمة بموجبها، على أن تعاد للمجلة خلال مدة أقصاها عشرة أيام.
- ✓ الأبحاث التي لم تتم الموافقة على نشرها لا تعاد إلى الباحثين.
- ✓ الأفكار الواردة فيما ينشر من دراسات وبحوث وعروض تعبر عن أراء أصحابها.
- ✓ لا يجوز نشر إي من المواد المنشورة في المجلة مرة أخرى.
- ✓ يدفع الراغب في نشر بحثه مبلغ قدره (400 د.ل) دينار ليبي إذا كان الباحث من داخل ليبيا، و (200 $) دولار أمريكي إذا كان الباحث من خارج ليبيا. علماً بأن حسابنا القابل للتحويل هو: **(بنغازي – ليبيا - مصرف التجارة والتنمية، الفرع الرئيسي - بنغازي، رقم 001-225540-0011. الاسم (صلاح الأمين عبدالله محمد)**.
- ✓ جميع المواد المنشورة في المجلة تخضع لقانون حقوق الملكية الفكرية للمجلة.

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General Form of Exact Distributions of Differences, Ratios and Products for "Bivariate Beta Distribution"

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Abstract

The beta distribution is one of the basic distributions that are applicable in different aspects. the beta distributions are used in extensively in Bayesian statistics. The bivariate beta distribution are used in a wide variety of applications such as Bayesian statistics and reliability theory. In this paper, we derive the exact distributions of the general form of $Z = \alpha X - \beta Y$, $R = {\alpha X}/{\beta Y}$ and $P = \alpha\beta XY$, and the corresponding moments , when X and Y follow the Bivariate beta distribution also called "Connor and Mosimann's generalized Dirichlet distribution". The calculations involve the use of special functions. Moreover, an application of the results to real data (Nadarajah and Kotz [2007]) is provided.

Key words: Bivariate beta distribution; Differences of random variables; Ratios of random variables; products of random variables; Hypergeometric function.

الملخص:

يعتبر توزيع بيتا من التوزيعات الهامة والذي له تطبيقات عديدة في مجالات مختلفة. كما ان توزيع بيتا له اهمية خاصة في الاحصاء البيزي. وكذلك توزيعات بيتا الثنائية تستخدم في كثير من التطبيقات، مثل الاحصاء البيزي ونظرية الموثوقية، لذا في هذا البحث نقوم باشتقاق التوزيعات التامة (المضبوطة) للاختلاف (الفرق) والنسبة وحاصل الضرب لمتغيرين عشوائيين يتبعان توزيع بيتا الثنائي (المعروف ايضا باسم توزيع ديريتشلت لكونور وموسيمان) كما اوجدنا دالة التوزيع التراكمي والعزوم لهذه التوزيعات. وفي نهاية البحث تم تطبيق النتائج النظرية التي توصلنا اليها على بيانات حقيقة.

1. Introduction.

The linear combinations, ratios and products of two random variables X and Y say have attracted researchers in the statistics literature, X and Y may be independent or dependent.

For independent variables, (Pham-Gia and Turkkan [1993]) derived the exact distribution of **sums**, differences of X and Yare independent beta random variables. Also, (Pham-Gia[2000]) has derived distributions for **products** of X and Yare independent beta random variables. For dependent variables, (Nadarajah [2005]) has derived distributions for the **sums, products, and** ratios for the bivarate Gumbel distribution, followed by (Nadarajah and Kotz [2007]) who derived the distributions for sums and ratios of Connor and Mosimann's generalized Dirichlet distribution. Also, Al-Ruzaiza and El-Gohary [2008] have derived distributions for the sums, products and ratios of inverted bivariate beta distribution. (Nadarajah [2005]) has also derived the reliability of some bivariate beta distribution.

The bivariate beta distribution is one of the basic distributions in statistics, as it attracted useful applications in several areas; for example, in the modeling of the proportions of substances in a mixture, brand shares, i.e the proportions of brands of some consumer product that are bought by customers (Chatfiled [1975]), proportions of the electorate voting for the candidate in a two candidate election [Hoyer and Mayer 1976] and the dependence between two soil strength parameters (A_Grivas and Asaoka [1982]). They have also been used extensively as a prior in Bayesian statistics (see, for example, Apostolakis [1987]).

In this paper, we derive the exact distributions of $Z = \alpha X - \beta Y$, $R = {\alpha X}/{\beta Y}$ and $P = \alpha\beta XY$, when X and Y are distributed according to the joint probability density function pdf given by

$$f(x,y) = \frac{\Gamma(a+c)\Gamma(b+d)}{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(d)} x^{a-1} y^{b-1} (1-x)^{c-b-d} (1-x-y)^{d-1} \quad (1)$$

$$for, x \ge 0, y \ge 0, x+y<1, a>0, b>0, d>0 \text{ and } c>0$$

The distribution in (1) is known as the Connor and Mosimann's generalized Dirichlet distribution (see Connor and Mosimann [1969]). It has several applications in many areas, including Bayesian statistics, contingency tables, correspondence analysis, environmental sciences, forensic sciences,geochemistry, image analysis and statistical decision theory (see, for example, Gupta and Nadarajah [2004] for illustrations of some of these application areas).

The calculations throughout this paper involve several special functions, including the incomplete beta function defined by

$$\mathrm{B}_x(\lambda,\mu) = \int_0^x t^{\lambda-1}(1-t)^{\mu-1}\,dt, \quad (2)$$

Which is given by

$$\mathrm{B}_x(\lambda,\mu) = \frac{x^\lambda}{\lambda}\, {}_2F_1(\lambda, 1-\mu; \lambda+1; x) \quad (3)$$

The Gauss hypergeometric function defined by

$$ {}_2F_1(\delta,\lambda;\lambda+\mu;x)=\frac{1}{B(\lambda,\mu)}\int_0^1 u^{\lambda-1}(1-u)^{\mu-1}(1-xu)^{-\delta}\,du \tag{4}$$

where $\operatorname{Re}\lambda>0,\ \operatorname{Re}\mu>0$, which is given in a series form by

$$ {}_2F_1(\delta,\lambda;\mu;x)=\sum_{j=0}^{\infty}\frac{(\delta)_j(\lambda)_j}{(\mu)_j}\frac{x^j}{j!}, \tag{5}$$

Where $|x|<1$, and $(f)_k=f(f+1)\ldots\ldots(f+k-1)$ denotes the ascending factorial, the Appell hypergeometric function of the first kind defined by

$$ F_1(\lambda,\delta,\gamma;\lambda+\mu;u,v)=\frac{1}{\mathrm{B}(\lambda,\mu)}\int_0^1 x^{\lambda-1}(1-x)^{\mu-1}(1-ux)^{-\delta}(1-vx)^{-\gamma}\,dx, \tag{6}$$

where $\operatorname{Re}\lambda>0,\ \operatorname{Re}\mu>0$, which is given in a series form by

$$ F_1(\lambda,\delta,\gamma;\mu;u,v)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\lambda)_{i+j}(\delta)_i(\gamma)_j u^i v^j}{(\mu)_{i+j}\, i!\, j!}, \tag{7}$$

where $|u|<1,\ |v|<1$.

The properties of the above special functions can be found in(Gragshteyn and Ryzhik [1980]).

This paper is organized as follows. Section2 deals with the derivation of probability density function (pdfs) and the cumulative distribution functions (cdfs) of $Z=\alpha X-\beta Y$, $R=\alpha X/\beta Y$ and $P=\alpha\beta XY$. The corresponding moments are discussed in Section3. Finally, Section4 provides an application to compositional data of lavas from Skye (see Nadarajah and Kotz [2007]).

2. Probability density functions.

In this section, we derive the exact probability density functions and cumulative distribution functions of general form of the differences, ratios and products of two random variables that follow "Connor and Mosimann's generalized Dirichlet distribution", given in (1)

Theorem1**:** If X and Y are jointly distributed random variables following the pdf given by Equation (1), then the pdf and cdf of $Z=\alpha X-\beta Y$ are given, respectively, by

$$ g(z)=\begin{cases}\dfrac{\Gamma(a+c)\Gamma(b+d)}{(\alpha+\beta)^a\beta^{b+d-1}\Gamma(b)\Gamma(c)\Gamma(a+d)}(-z)^{b-1}(\beta+z)^{a+d-1}F_1\left(a,b+d-c,1-b;a+d;\dfrac{\beta+z}{\alpha+\beta},\dfrac{\alpha(\beta+z)}{z(\alpha+\beta)}\right) & if\ \ -\beta<z<0\\[2ex] \dfrac{\alpha^{b-1}\beta^{a-1}\mathrm{B}(a+b-1,d)}{(\alpha+\beta)^{a+b-1}\mathrm{B}(a,c)\mathrm{B}(b,d)^2}F_1\left(a+b-1,b+d-c;a+b+d-1;\dfrac{\beta}{\alpha+\beta}\right) & if\ \ z=0\\[2ex] \dfrac{1}{(\alpha+\beta)^b\alpha^{a+c-b-1}\mathrm{B}(a,c)}z^{a-1}(\alpha-z)^{c-1}F_1\left(b,b+d-c,1-a;b+d;\dfrac{\beta}{\alpha+\beta},\dfrac{\beta(z-\alpha)}{z(\alpha+\beta)}\right) & if\ \ 0<z<\alpha\\ 0 & otherwise\end{cases} \tag{8}$$

Where $a+b>1$

and

$$G(z)=\begin{cases} 0 & if \quad z\le-\beta \\ \dfrac{\Gamma(a+c)\Gamma(b+d)}{(\alpha+\beta)^{a}\beta^{b+d-1}\Gamma(b)\Gamma(c)\Gamma(a+d)}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\dfrac{(a)_{m+n}(b+d-c)_{m}(1-b)_{n}}{(a+d)_{m+n}m!n!}\dfrac{\alpha^{n}\beta^{a+d+b+m-1}}{(\alpha+\beta)^{m+n}}(-1)^{n}\mathrm{B}_{\left(\frac{\beta+z}{\beta}\right)}(a+d+m+n,b-n) & if \; -\beta<z<0 \\ \dfrac{\Gamma(a+c)\Gamma(b+d)}{(\alpha+\beta)^{a}\beta^{b+d-1}\Gamma(b)\Gamma(c)\Gamma(a+d)}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\dfrac{(a)_{m+n}(b+d-c)_{m}(1-b)_{n}}{(a+d)_{m+n}m!n!}\dfrac{\alpha^{n}\beta^{a+b+d+m-1}}{(\alpha+\beta)^{m+n}}(-1)^{n}\beta^{a+b+d+m-1}\mathrm{B}(a+d+m+n,b-n) & \\ +\dfrac{1}{(\alpha+\beta)^{b}\alpha^{a+c-b-1}\mathrm{B}(a,c)}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\dfrac{(b)_{m+n}(b+d-c)_{m}(1-a)_{n}}{(b+d)_{m+n}m!n!}\dfrac{\beta^{m}\alpha^{a+c-1}}{(\alpha+\beta)^{m+n}}(-1)^{n}\mathrm{B}_{\frac{z}{\alpha}}(a-n,c+n) & if \quad 0\le z<1 \\ 1 & if \quad z\ge1 \end{cases}$$

(9)

Proof: From Equation (1), the joint pdf of $Z=\alpha X-\beta Y$ and Y is given by

$$f(z,y)=\frac{\Gamma(a+c)\Gamma(b+d)}{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(d)}\left(\frac{z+\beta y}{\alpha}\right)^{a-1}y^{b-1}\left(1-\frac{z+\beta y}{\alpha}\right)^{c-b-d}\left(1-\frac{z+(\alpha+\beta)y}{\alpha}\right)^{d-1},$$

$$-\beta<z<\alpha,\quad \frac{-z}{\beta}<y<\frac{\alpha-z}{\alpha+\beta}.$$

Thus, the marginal pdf of Z is

$$g(z)=\int_{-\infty}^{\infty}f(z,y)dy,\qquad -\beta<z<\alpha.$$

For $-\beta<z<0$, the pdf of Z is given by

$$g(z)=\frac{\Gamma(a+c)\Gamma(b+d)}{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(d)}\frac{1}{\alpha}\int_{-\frac{z}{\beta}}^{\frac{\alpha-z}{\alpha+\beta}}y^{b-1}\left(\frac{z+\beta y}{\alpha}\right)^{a-1}\left(1-\frac{z+\beta y}{\alpha}\right)^{c-b-d}\left(1-\frac{z+(\alpha+\beta)y}{\alpha}\right)^{d-1}dy$$

$$=\frac{\Gamma(a+c)\Gamma(b+d)}{\alpha^{a}\beta^{b+d-1}\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(d)}(-z)^{b-1}(\beta+z)^{d-1}\int_{0}^{1}\left(\frac{\alpha(\beta+z)}{\alpha+\beta}v\right)^{a-1}\left(1-\frac{\beta+z}{\alpha+\beta}v\right)^{c-b-d}\left(1-\frac{\alpha(\beta+z)}{(\alpha+\beta)z}v\right)^{b-1}(1-v)^{d-1}\frac{\alpha(\beta+z)}{\alpha+\beta}dv$$

where the transformations $v=\frac{(\alpha+\beta)(z+\beta y)}{\alpha(\beta+z)}$ have been applied. Solving above integral using (6), we get the first part of Equation (8).

Now, for $z=0$, we have

$$g(z)=\frac{\Gamma(a+c)\Gamma(b+d)}{\alpha^{a}\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(d)}(\beta)^{a-1}\int_{0}^{\frac{\alpha}{\alpha+\beta}}y^{a+b-2}\left(1-\frac{\beta}{\alpha}y\right)^{c-b-d}\left(1-\frac{(\alpha+\beta)}{\alpha}y\right)^{d-1}dy$$

$$=\frac{\alpha^{c-d-1}\beta^{a-1}\Gamma(a+c)\Gamma(b+d)}{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(d)}\left(\frac{1}{\alpha+\beta}\right)^{a+c-d-1}\int_{0}^{1}v^{d-1}(1-v)^{a+b-2}\left(1+\frac{\beta}{\alpha}v\right)^{-(b+d-c)}dv$$

Using the transformation $v=\left(1-\frac{(\alpha+\beta)}{\alpha}y\right)$ and(4), we obtain the second part of Equation (8). As for $0\le z<\alpha$, we have

$$=\frac{\Gamma(a+c)\Gamma(b+d)}{\alpha^{a+c-b-1}\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(d)}\int_0^{\frac{\alpha-z}{\alpha+\beta}} y^{b-1}\left(z\left(1+\frac{\beta}{z}y\right)\right)^{a-1}\left((\alpha-z)\left[1-\frac{\beta}{\alpha-z}y\right]\right)^{c-b-d}\left((\alpha-z)\left[1-\frac{(\alpha+\beta)}{\alpha-z}y\right]\right)^{d-1}dy$$

$$=\frac{\Gamma(a+c)\Gamma(b+d)}{(\alpha+\beta)^b\alpha^{a+c-b-1}\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(d)}z^{a-1}(\alpha-z)^{c-1}\int_0^1 u^{b-1}(1-u)^{d-1}\left(1-\frac{\beta}{(\alpha+\beta)}u\right)^{-(b+d-c)}\left(1-\frac{\beta(z-\alpha)}{(\alpha+\beta)z}u\right)^{-(1-a)}du$$

where the transformation $u=\frac{(\alpha+\beta)y}{\alpha-z}$ has been applied. Solving above integral using (6), we get the last part of Equation (8).

Integration of $g(z)$ in (8) with respect to Z and using (7), leads to Equation (9), which completes the proof.

The different plots of "Figure 1" illustrate the shapes of the pdf of Z for selected values of α,β,a,b,c and d, where it is obvious that the effect of the parameters is evident.

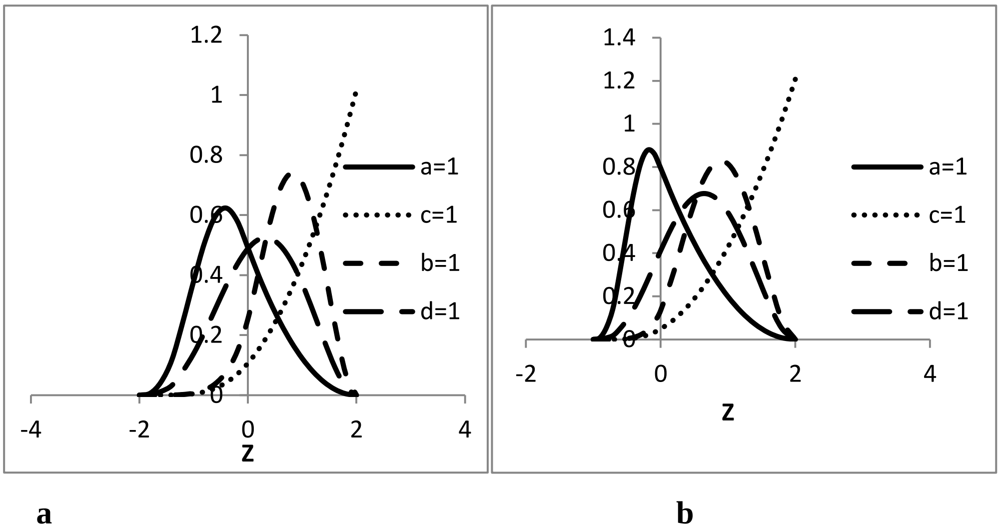


Fig. 1 Plots of the pdf of $Z=\alpha X-\beta Y$ for (a): $\alpha=2,\beta=2$; when a=1 (b=c=d=3) ,when b=1 (a=c=d =3) ,when c=1 (a=b=d=3),when d=1 (a=b =c=3). (b): $\alpha=2,\beta=1$; when a=1 (b=c=d=3) ,when b=1 (a=c=d =3) ,when c=1 (a=b=d=3),when d=1 (a=b =c=3).

Theorem 2: If X and Y are jointly distributed random variables with the joint pdf (1) then the pdf and cdf of $R=\alpha X/\beta Y$ are given, respectively, by

$$g(r)=\frac{\Gamma(a+c)\Gamma(a+b)\Gamma(b+d)}{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(a+b+d)}\left(\frac{\beta}{\alpha}\right)^a\frac{r^{a-1}}{\left(1+\frac{\beta}{\alpha}r\right)^{a+b}}{}_2F_1\left(b+d-c,a+b;a+b+d;\frac{\frac{\beta}{\alpha}r}{1+\frac{\beta}{\alpha}r}\right) \qquad (10)$$

Where $0 < r < \infty$, and

$$G(r) = \frac{\Gamma(a+c)\Gamma(a+b)\Gamma(b+d)}{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(a+b+d)}\left(\frac{\beta}{\alpha}\right)^{a}\sum_{l=0}^{\infty}\frac{(a+b)_l(b+d-c)_l}{(a+b+d)_l\, l!}B_{(r/(1+r))}(a+l,b)$$

(11)

Proof: Using (1), the joint pdf of $R = \alpha X/\beta Y$ and Y is given by

$$f(r,y) = \frac{1}{B(a,c)B(b,d)}\left(\frac{\beta}{\alpha}ry\right)^{a-1} y^{b-1}\left(1-\frac{\beta}{\alpha}ry\right)^{c-b-d}\left(1-\frac{\beta}{\alpha}ry-y\right)^{d-1}, 0 < r < \infty, 0 < \left(1+\frac{\beta}{\alpha}r\right)y < 1$$

Thus, the marginal pdf of R is

$$g(r) = \int_{-\infty}^{\infty} y f(r,y)dy$$

$$= \frac{1}{B(a,c)B(b,d)}\left(\frac{\beta}{\alpha}\right)^{a-1}\frac{r^{a-1}}{\left(1+\frac{\beta}{\alpha}r\right)^{a+b}}\int_0^1 u^{a+b-1}(1-u)^{d-1}\left(1-\frac{\frac{\beta}{\alpha}r}{\alpha\left(1+\frac{\beta}{\alpha}r\right)}u\right)^{-(b+d-c)} du$$

where the transformation $u = \left(1+\frac{\beta}{\alpha}r\right)y$. Using (4) we obtain (10).

Equation (11) is obtained by integration of $g(r)$ in (10), with respect to r. completes the proof of the theorem.

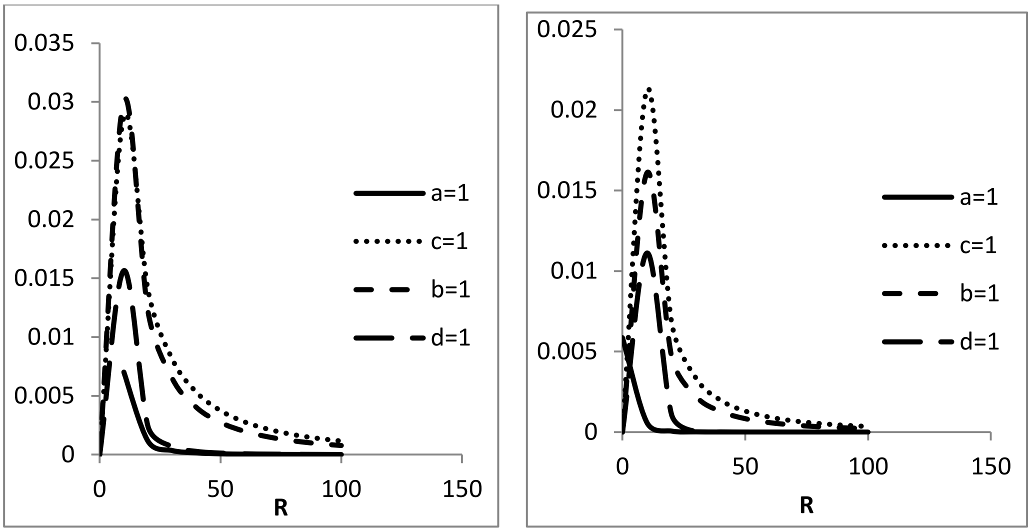
Remark:

The pdf of $R = \alpha X/\beta Y$ when $\alpha = \beta$ is given by

$$g(r) = \frac{\Gamma(a+c)\Gamma(a+b)\Gamma(b+d)}{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(a+b+d)}\frac{r^{a-1}}{(1+r)^{a+b}}\,{}_2F_1\left(b+d-c, a+b; a+b+d; \frac{r}{1+r}\right)$$

see (N.A.Mokhlis etal. [2013])

Figure 2 with its variants plots illustrates the shapes of the pdf of R for selected values of α, β, a, b, c and d where it is obvious that the effect of the parameters is evident.



a b

Fig. 2 Plots of the pdf of $R=\alpha X/\beta Y$ for (a): $\alpha=1,\beta=2$ when a=1 (b=c=d=3) ,when b=1 (a=c=d=3) ,when c=1 (a=b=d=3),when d=1 (a=b=c=3). (b) : $\alpha=2,\beta=1$ when a=1 (b=c=d=3) ,when b=1 (a=c=d=3) ,when c=1 (a=b=d=3),when d=1 (a=b=c=3).

Theorem 3: If X and Y are jointly distributed random variables with the joint pdf (1) then the pdf and cdf of $P=\alpha\beta XY$ are given, respectively, by

$$g_P(p)=\frac{\Gamma(a+c)\Gamma(b+d)\Gamma(d)}{2^{a+c-2b-2d}(\alpha\beta)^b\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(2d)}p^{b-1}\left(1-\sqrt{1-4\frac{p}{\alpha\beta}}\right)^{a-b-d}\left(1+\sqrt{1-\frac{4p}{\alpha\beta}}\right)^{c-b-d}$$

$$\times\left(1-\frac{4}{\alpha\beta}p\right)^{d-\frac{1}{2}}F_1\left(d,b+d-a,b+d-c;2d;2-\frac{\alpha\beta+\sqrt{(\alpha\beta)^2-4\alpha\beta p}}{2p},2-\frac{\alpha\beta-\sqrt{(\alpha\beta)^2-4\alpha\beta p}}{2p}\right) \tag{12}$$

where $0<p<\frac{\alpha\beta}{4}$

and

$$G(p)=T\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(d)_{m+n}(b+d-a)_m(b+d-c)_n}{(2d)_{m+n}m!n!}\left(\frac{1}{2}\right)^{m+n}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{m+i}\binom{a-b-d+n}{i}$$

$$\times\binom{c-b-d+m}{j}\left(\frac{4p}{\alpha\beta}\right)^{(m+n-b)}\mathrm{B}_{\frac{4p}{\alpha\beta}}\left(b-m-n,d+\frac{1}{2}(m+n+i+j+1)\right) \tag{13}$$

where $T=\frac{\Gamma(a+c)\Gamma(b+d)\Gamma(d)}{2^{a+c-2b-2d}(\alpha\beta)^b\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(2d)}$

Proof: From Equation (1), the joint pdf of X and $P=\alpha\beta XY$ is given by

$$f(x,p)=Kx^{a-1}\left(\frac{p}{\alpha\beta x}\right)^{b-1}(1-x)^{c-b-d}\left(1-x-\frac{p}{\alpha\beta x}\right)^{d-1}\left(\frac{1}{\alpha\beta x}\right),\quad 0<p<\frac{\alpha\beta}{4}$$

$$=K\left(\frac{1}{\alpha\beta}\right)^{b}p^{b-1}x^{a-b-d}(1-x)^{c-b-d}(x-p_1)^{d-1}(p_2-x)^{d-1}$$

where $K=\frac{\Gamma(a+c)\Gamma(b+d)}{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(d)}$, $p_1=\frac{\left(1-\sqrt{1-\frac{4p}{\alpha\beta}}\right)}{2}$ and $p_2=\frac{\left(1+\sqrt{1-\frac{4p}{\alpha\beta}}\right)}{2}$.

Thus, the marginal pdf of P can be written as

$$f(p)=\frac{Kp^{b-1}}{(\alpha\beta)^{b}}\int_{p_1}^{p_2}x^{a-b-d}(1-x)^{c-b-d}(x-p_1)^{d-1}(p_2-x)^{d-1}dx.$$

$$=\frac{Kp^{b-1}}{(\alpha\beta)^{b}}p_1^{a-b-d}(1-p_1)^{c-b-d}(p_2-p_1)^{2d-1}\int_0^1 v^{d-1}(1-v)^{d-1}\left(1-\left(1-\frac{p_2}{p_1}\right)v\right)^{-(b+d-a)}$$

$$\times\left(1-\frac{p_2-p_1}{1-p_1}v\right)^{-(b+d-c)}dv.$$

Using the transformations $v=\frac{x-p_1}{p_2-p_1}$ and (6) , we have

$$f(p)=\frac{Kp^{b-1}}{(\alpha\beta)^{b}}p_1^{a-b-d}(1-p_1)^{c-b-d}(p_2-p_1)^{2d-1}\mathrm{B}(d,d).$$

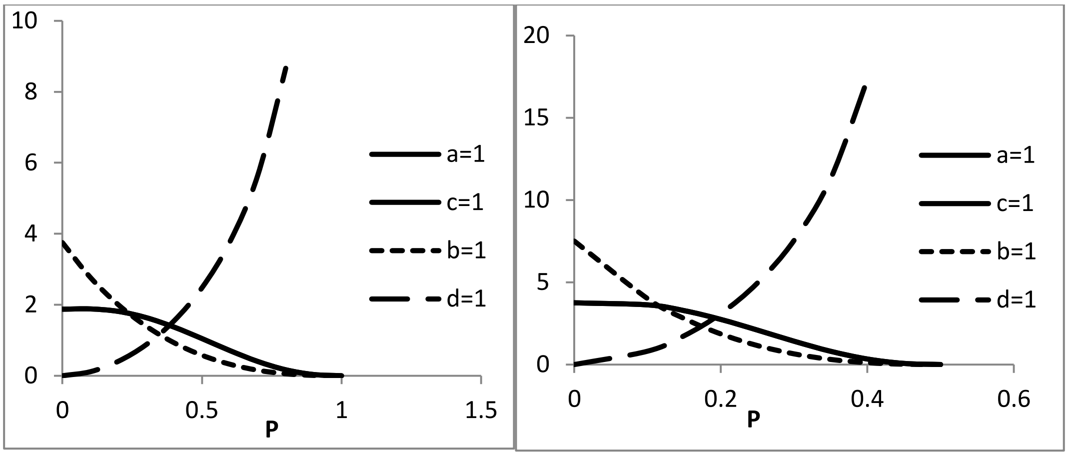
$$\times F_1\left(d,b+d-a,b+d-c;2d;1-\frac{p_2}{p_1},\frac{p_2-p_1}{1-p_1}\right)$$

Hence the pdf can be rewritten as the result given in Equation (12). To obtain the cdf of P, integration of $f(p)$ in (12) with respect to P and using (7), leads to the result (13), which completes the proof.

Remark:

When $\alpha=1,\beta=2$ or $\alpha=2,\beta=1$.Then pdf of $P=\alpha\beta XY$ gives the same results.

Figure 3 illustrates the shapes of the pdf of P for selected values of α,β,a,b,c and d, of course, it is obvious that the effect of the parameters is evident.



A b

Fig. 3 Plots of the pdf of $P=\alpha\beta XY$ for (a): $\alpha=2,\beta=2$, when a= 1 (b=c=d=3), b= 1 (a=c=d=3), when c=1 (a=b=d=3), when d=1 (a=b=c=3). (b): $\alpha=2,\beta=1$, when a= 1 (b=c=d=3), b= 1 (a=c=d=3), when c=1 (a=b=d=3), when d=1 (a=b=c=3).

Remark: It is evident that there is similarity in the curves of plots when a=1,c=1.

3. Moments

For deriving the moments of Z, R and P, we need the following lemma:

Lemma1 If X and Y are jointly distributed random variables with the joint pdf in (1), then

$$\mathrm{E}\left(X^{n}Y^{m}\right)=K\mathrm{B}(a+n,c+m)\mathrm{B}(b+m,d) \tag{14}$$

for $a+n>0, b+m>0, c+m>0$ and $K=\dfrac{1}{B(a,c)B(b,d)}$

Proof: Knowing that

$$\mathrm{E}\left(X^{n}Y^{m}\right)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}X^{n}Y^{m}f(x,y)dydx, \tag{15}$$

and substituting with (1) into (15), we get

$$\mathrm{E}\left(X^{n}Y^{m}\right)=K\int_{0}^{1}\int_{0}^{1-x}x^{n+a-1}y^{m+b-1}(1-x)^{c-b-d}\ (1-x-y)^{d-1}dydx$$

$$=K\int_{0}^{1}x^{n+a-1}(1-x)^{c-b-d}\left(\int_{0}^{1-x}y^{m+b-1}[(1-x)\left(1-\frac{y}{1-x}\right)^{d-1}]dy\right)dx$$

Using the transformation $u=\dfrac{y}{1-x}$, we obtain

$$\mathrm{E}\left(X^{n}Y^{m}\right)=K\int_{0}^{1}x^{n+a-1}(1-x)^{c-b-1}\left(\int_{0}^{1}[u(1-x)]^{m+b-1}(1-u)^{d-1}(1-x)du\right)dx$$

$$=K\int_{0}^{1}x^{n+a-1}(1-x)^{c+m-1}dx\int_{0}^{1}u^{m+b-1}(1-u)^{d-1}du,$$

Solving the integral, we obtain the result (14). This completes the proof of the lemma.

Theorem4: If X and Y are jointly distributed according to the joint pdf (1), and $Z=\alpha X-\beta Y$, then

$$\mathrm{E}(Z^m)=K\sum_{l=0}^{m}(-1)^{m-l}\binom{m}{l}\alpha^m\beta^{m-l}\mathrm{B}(a+l,c+m-l)\mathrm{B}(b+m-l,d) \qquad (16)$$

for $m>1,\ b+m>l, c+m>l,$ and $K=\dfrac{1}{B(a,c)B(b,d)}$

Proof: Starting with the definition of expectation and using the binomial expansion, we get

$$\mathrm{E}(Z^m)=\mathrm{E}[(\alpha X-\beta Y)^m]=\sum_{l=0}^{m}(-1)^{m-l}\binom{m}{l}\alpha^m\beta^{m-l}\mathrm{E}(X^lY^{m-l}),\quad m\geq 1.$$

Applying (14), we obtain the result (16).

Note that, if m=1, then

$$\mathrm{E}(Z)=\frac{\alpha a(b+d)-\alpha\beta cb}{(a+c)(b+d)}$$

Also, if m=2, then

$$\mathrm{E}(Z^2)=\frac{\alpha^2\beta^2cb(c+1)(b+1)-2\alpha^2\beta acb(b+d+1)+\alpha^2a(a+1)(b+d)(b+d+1)}{(a+c+1)(a+c)(b+d+1)(b+d)}$$

and hence

$$Var(Z)=\frac{\alpha^2\beta^2cb(c+1)(b+1)-2\alpha^2\beta acb(b+d+1)+\alpha^2a(a+1)(b+d)(b+d+1)}{(a+c+1)(a+c)(b+d+1)(b+d)}-\left(\frac{\alpha a(b+d)-\alpha\beta cb}{(a+c)(b+d)}\right)^2$$

Theorem5: If X and Y are jointly distributed according to the joint pdf (1), and $R={\alpha X}/{\beta Y}$, then

$$\mathrm{E}(R^m)=K\left(\frac{\alpha}{\beta}\right)^m\mathrm{B}(a+m,c-m)\mathrm{B}(b-m,d) \qquad (17)$$

for $m\geq 1,\ b>m,\ c>m,$ and $K=\dfrac{1}{B(a,c)B(b,d)}$

Proof: writing $\mathrm{E}(R^m)=\mathrm{E}\left(\left(\frac{\alpha}{\beta}\right)^m X^mY^{-m}\right)$ and substituting in Equation (14), we obtain the result (17), and this completes the proof.

Now, if m=1

$$\mathrm{E}(R)=\left(\frac{\alpha}{\beta}\right)\frac{a(b+d-1)}{(c-1)(b-1)}$$

$b>1,\ c>1.$ where

Also, if m=2, then

$$\mathrm{E}\left(R^2\right)=\left(\frac{\alpha}{\beta}\right)^2\frac{a(a+1)(b+d-1)(b+d-2)}{(c-1)(c-2)(b-1)(b-2)}$$

and hence

$$Var(R)=\left(\frac{\alpha}{\beta}\right)^2\left(\frac{a(a+1)(b+d-1)(b+d-2)}{(c-1)(c-2)(b-1)(b-2)}-\left(\frac{a(b+d-1)}{(c-1)(b-1)}\right)^2\right)$$

where $b>2,\ c>2.$

Theorem6: If X and Y are jointly distributed according to the joint pdf (1), and P=αβXY, then

$$\mathrm{E}\left(P^m\right)=K(\alpha\beta)^m\,\mathrm{B}(a+m,c+m)\mathrm{B}(b+m,d) \tag{18}$$

Proof: Setting $m=n$ in the relation (14), we get the result (18), thus completing the proof.

Now, if m=1, then

$$\mathrm{E}(P)=\frac{\alpha\beta acb}{(a+c+1)(a+c)(b+d)}$$

Also, if m=2, then

$$\mathrm{E}\left(P^2\right)=\frac{(\alpha\beta)^2\,acb(a+1)(c+1)(b+1)}{(a+c+3)(a+c+2)(a+c+1)(a+c)(b+d+1)(b+d)}$$

and hence

$$Var(P)=(\alpha\beta)^2\left(\frac{acb(a+1)(c+1)(b+1)}{(a+c+3)(a+c+2)(a+c+1)(a+c)(b+d+1)(b+d)}-\left(\frac{acb}{(a+c+1)(a+c)(b+d)}\right)^2\right)$$

4. Application

Here, we provide an application of the results in Sections 2 and3. We use the data set on compositions of lavas from Skye (see Nadarajah and Kotz [2007]). The three variables are: A = sodium and potassium oxides, F = iron oxide and M = magnesium oxide.

Note that each column in Table 1 add to 1.Our interest is showing the differences, ratios and products of proportions of A and F, the proportions of A and M , and the proportions of F and M. An obvious model in this situation would be the generalized Dirichlet distribution given by the joint pdf (1).

A	F	M
0.52	0.42	0.06
0.52	0.44	0.05
0.47	0.48	0.05
0.45	0.49	0.06
0.4	0.5	0.1
0.37	0.54	0.09
0.27	0.58	0.15
0.27	0.54	0.19
0.23	0.59	0.18
0.22	0.59	0.19
0.21	0.6	0.19
0.25	0.53	0.22
0.24	0.54	0.22
0.22	0.55	0.23
0.22	0.56	0.22
0.2	0.58	0.22
0.16	0.62	0.22
0.17	0.57	0.26
0.14	0.54	0.32
0.13	0.55	0.32
0.13	0.52	0.35
0.14	0.47	0.39
0.24	0.56	0.2

Table 1. Data on compositions of lavas from Skye.

The distribution in (1) reasonably fits the three bivariate data sets on proportions: data set 1 containing the values (A, F), data set 2 containing the values (A, M), and data set 3 containing the values (F, M).Table2 gives the estimates of a, b, c and d, which were obtained using the maximum likelihood method (see Nadarajah and Kotz [2007]). Table 3 gives estimated values of the moments $\mathrm{E}(Z)$, $\mathrm{E}(R)$ and $\mathrm{E}(P)$ with different value of α, β obtained using (16), (17) and (18) respectively.

Data set	$\hat{a}$	$\hat{b}$	$\hat{c}$	$\hat{d}$
(A, F)	3.827	13.474	10.350	4.497
(A, M)	3.827	4.497	10.350	13.474
(F, M)	53.331	1.991	45.930	2.724

Table 2 Estimated values of a, b, c **and** d.

Data set	(A, F)	(A, M)	(F, M)
$\alpha = 2, \beta = 2$ $E(Z)$	-1.6496	-0.1909	0.2930
$\alpha = 2, \beta = 1$ $E(Z)$	-0.5548	0.1745	0.6838
$\alpha = 1, \beta = 2$ $E(Z)$	-0.8248	-0.0954	0.1465
$\alpha = 2, \beta = 2$ $E(R)$	0.5569	1.9864	4.4497
$\alpha = 2, \beta = 1$ $E(R)$	1.1137	3.9727	8.8994
$\alpha = 1, \beta = 2$ $E(R)$	0.2784	0.9932	2.2248
$\alpha = 2, \beta = 2$ $E(P)$	0.5521	0.1843	0.4157
$\alpha = 2, \beta = 1$ $E(P)$	0.2760	0.0921	0.2079
$\alpha = 1, \beta = 2$ $E(P)$	0.2760	0.0921	0.2079

Table3 Estimated $E(Z)$, $E(R)$ **and** $E(P)$.

We see that results in Table (3) are consistent with graphs in Figures 4-12, 13-18 and 19-24 .Show the fitted pdfs of Z given by (8), of R given by (10), of P given by (12) using the estimated parameters and using data.

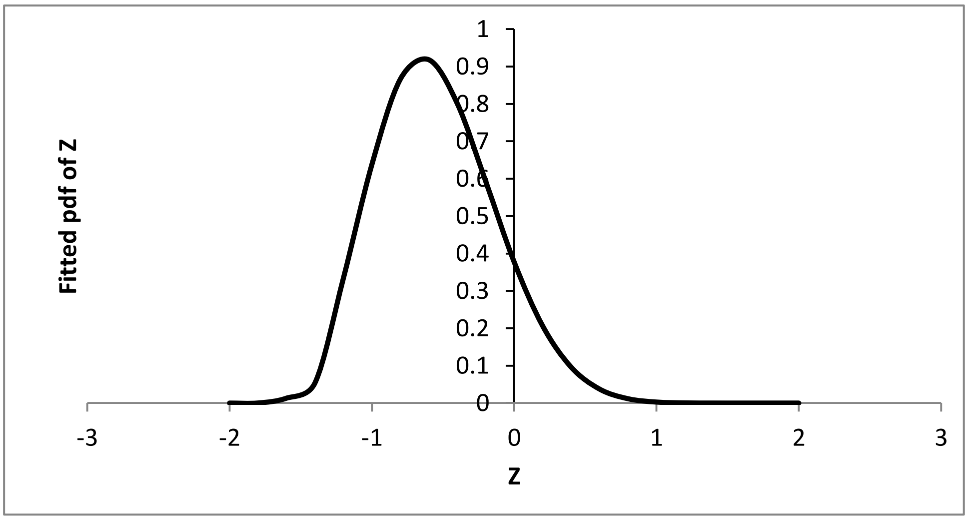


Fig. 4 Fitted pdf of Z= αX -βY given by (8) when α=2,β=2 and X is sodium and potassium oxides and Y is iron oxide.

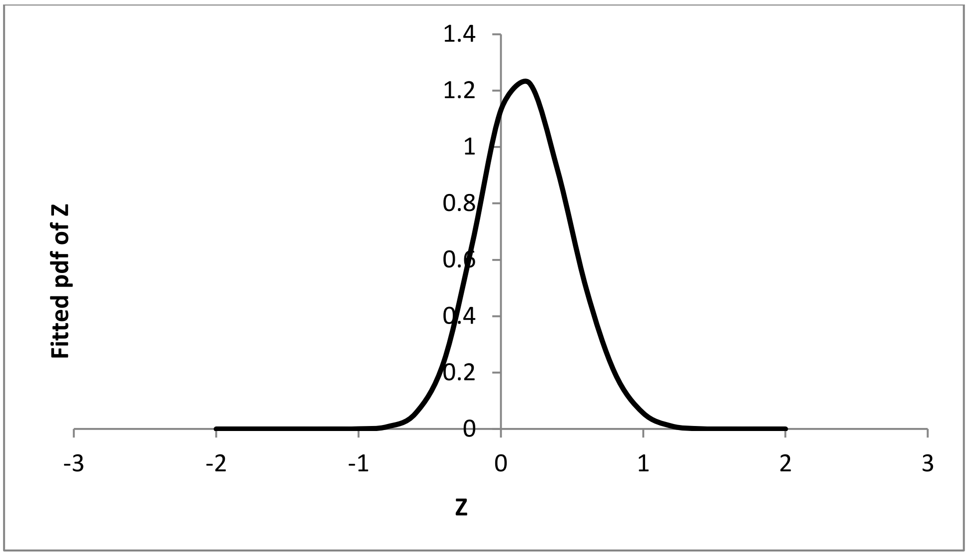


Fig. 5 Fitted pdf of Z= αX -βY given by (8) when α=2,β=2 and X is sodium and potassium oxides and Y is magnesium

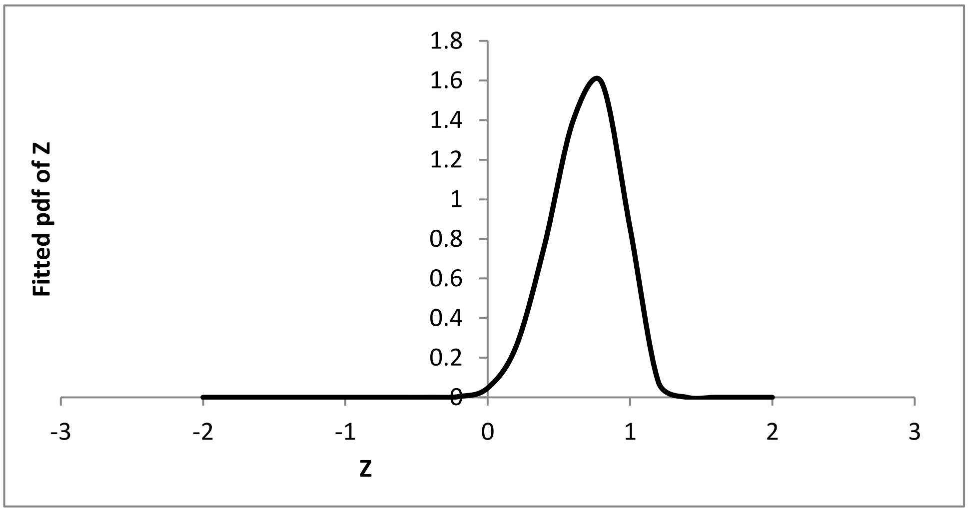


Fig. 6 Fitted pdf of Z= αX -βY given by (8) when α=2,β=2 and X is iron oxide and Y is magnesium oxide

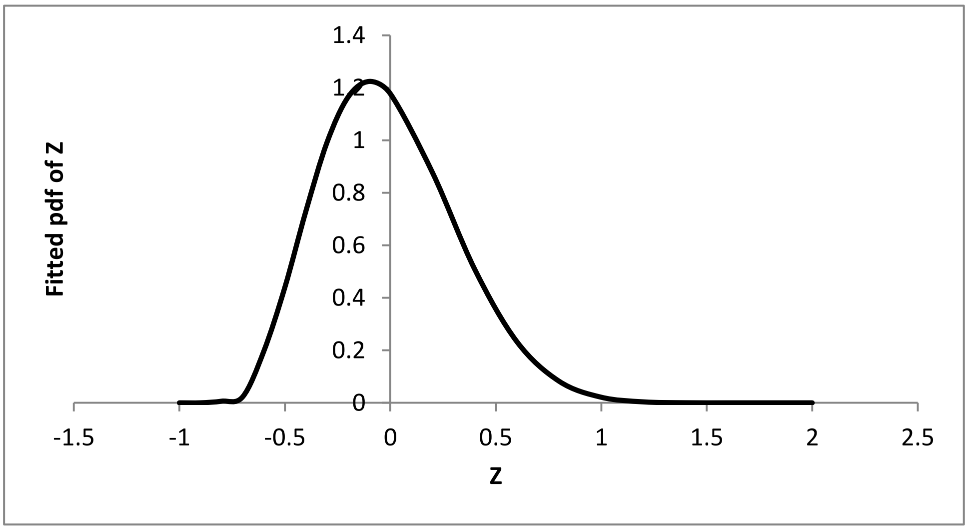


Fig. 7 Fitted pdf of Z= αX -βY given by (8) when α=2,β=1 and *X* is sodium and potassium oxides and *Y* is iron oxide.

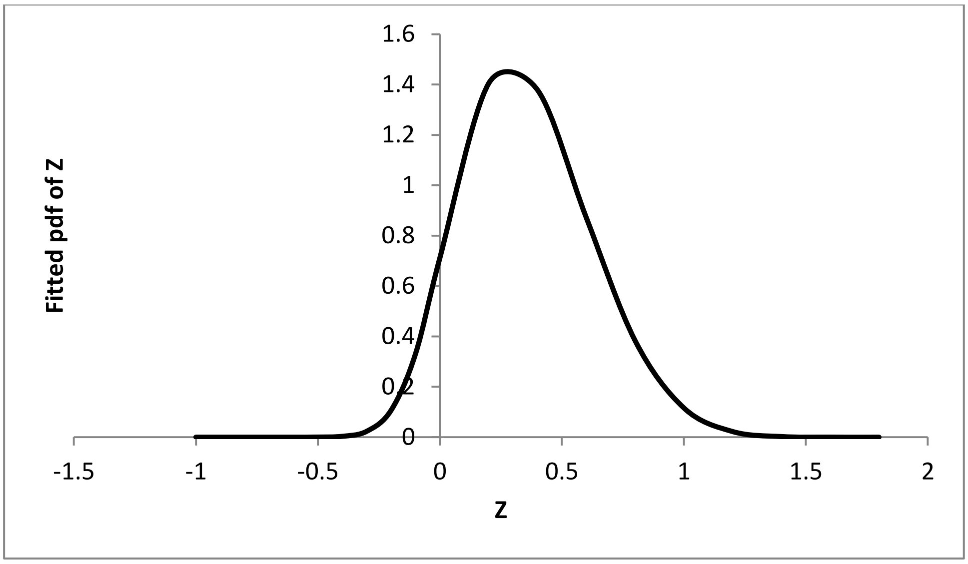


Fig. 8 Fitted pdf of Z= αX -βY given by (8) when α=2,β=1 and X is sodium and potassium oxides and Y is magnesium

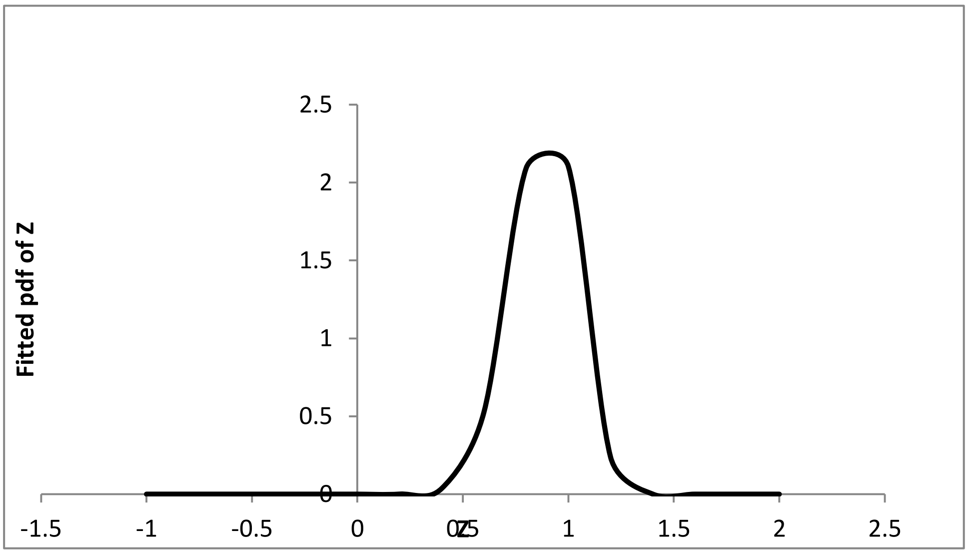


Fig. 9 Fitted pdf of Z= αX -βY given by (8) when α=2,β=1 and X is iron oxide and Y is magnesium oxide

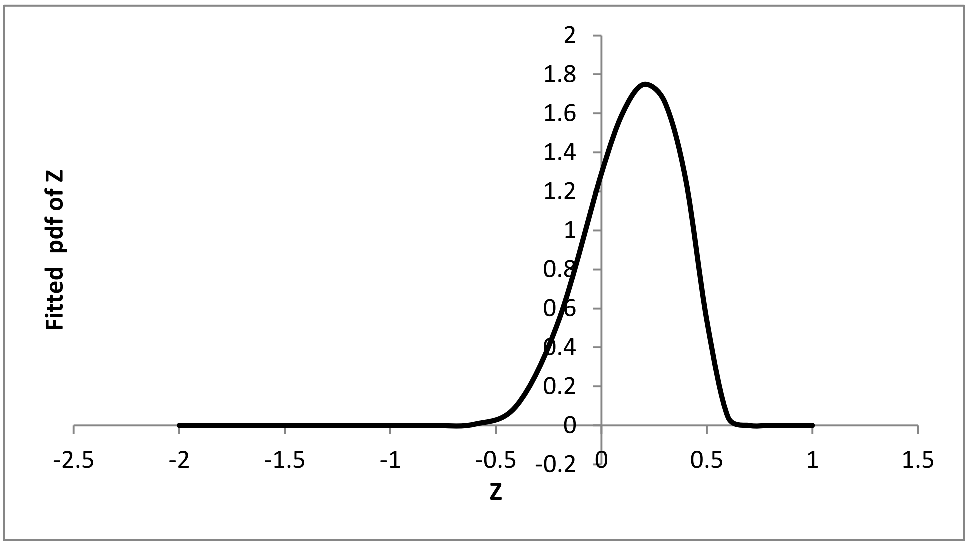


Fig. 10 Fitted pdf of Z= αX -βY given by (8) when α=1,β=2 and X is sodium and potassium oxides and Y is iron oxide.

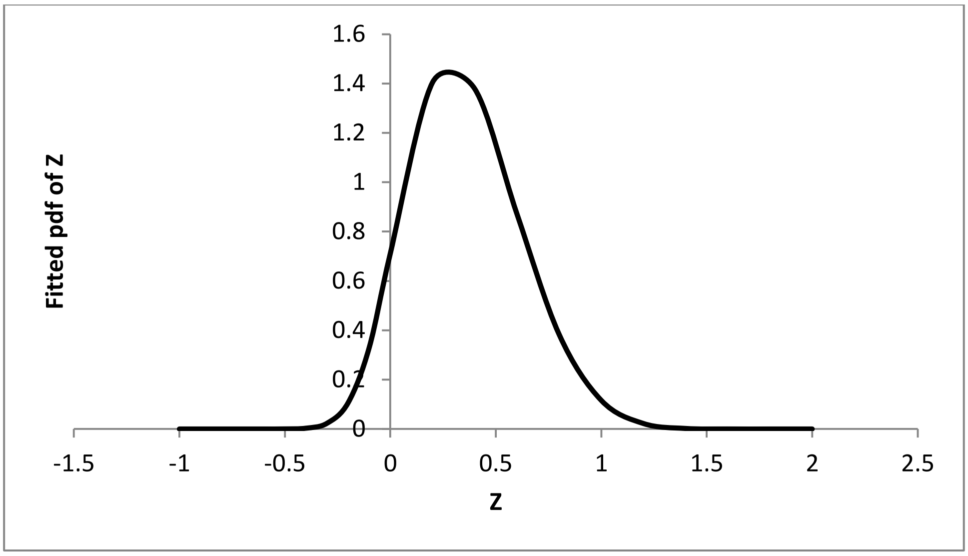


Fig. 11 Fitted pdf of Z= αX -βY given by (8) when α=1,β=2 and X is sodium and potassium oxides and Y is magnesium

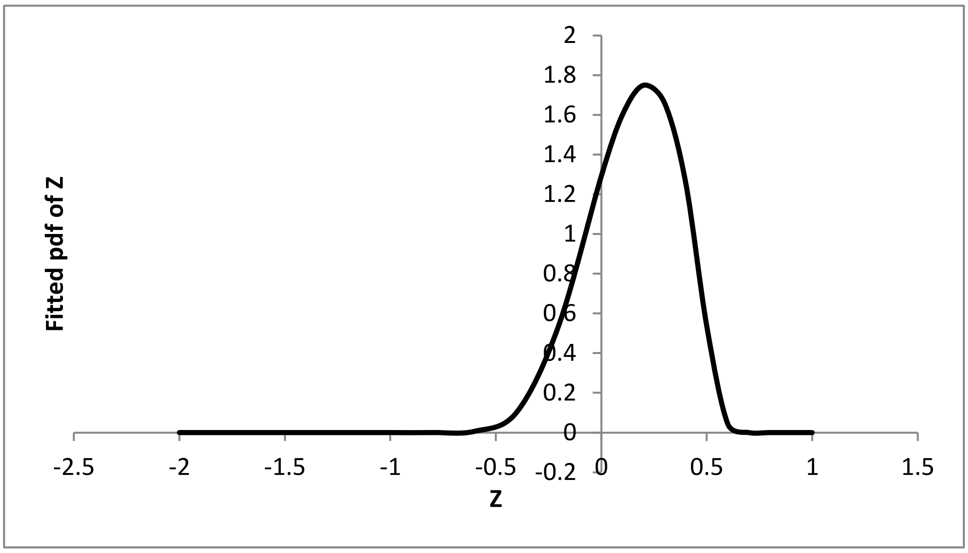


Fig. 12 Fitted pdf of Z= αX -βY given by (8) when α=1,β=2 and X is iron oxide and Y is magnesium oxide

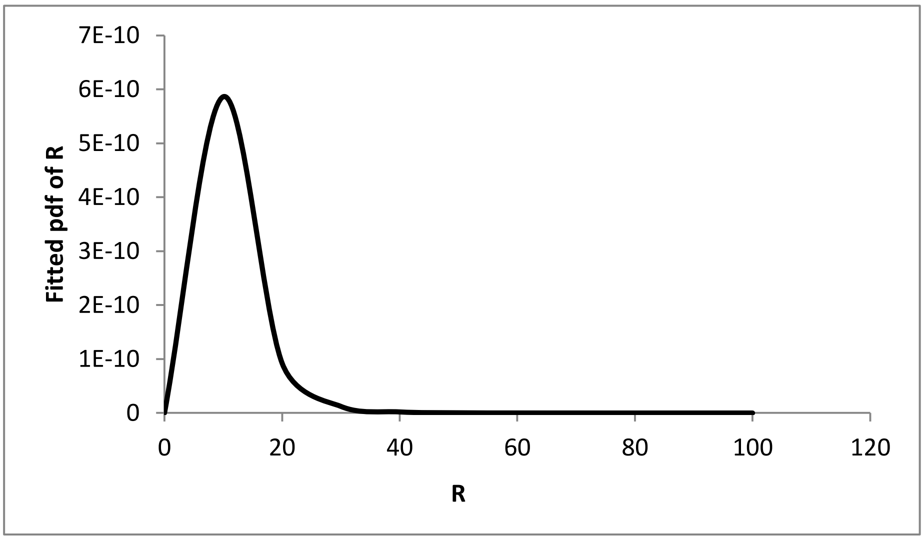


Fig. 13 Fitted pdf of R= $\alpha X/\beta Y$ given by (10) when α=2,β=1 and X is sodium and potassium oxides and Y is iron oxide.

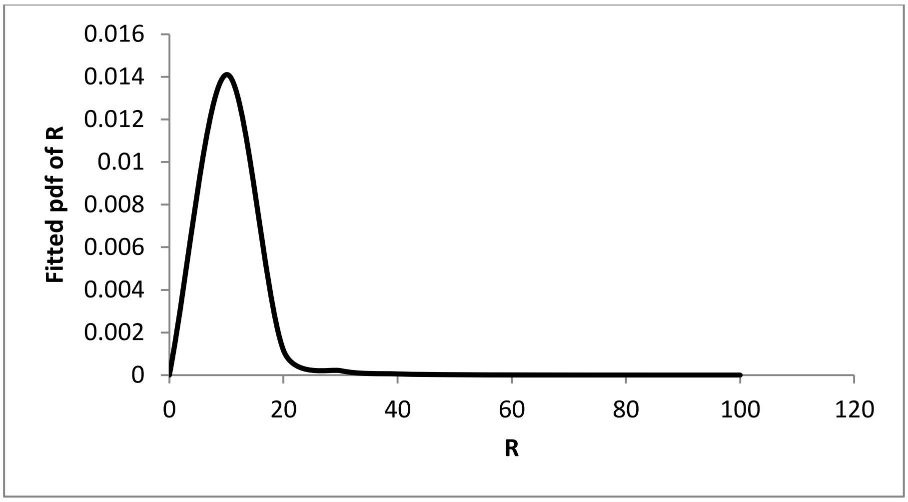


Fig. 14 Fitted pdf of R= $\alpha X/\beta Y$ given by (10) when α=2,β=1 and X is sodium and potassium oxides and Y is magnesium oxide.

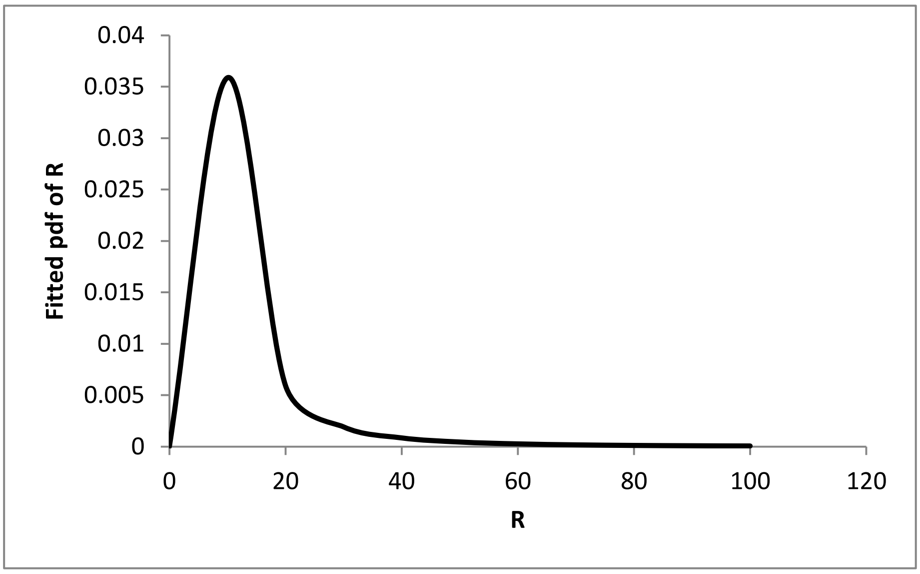


Fig. 15 Fitted pdf of R= $\alpha X/\beta Y$ given by (10) when α=2,β=1 and X is iron oxide and Y is magnesium oxide.

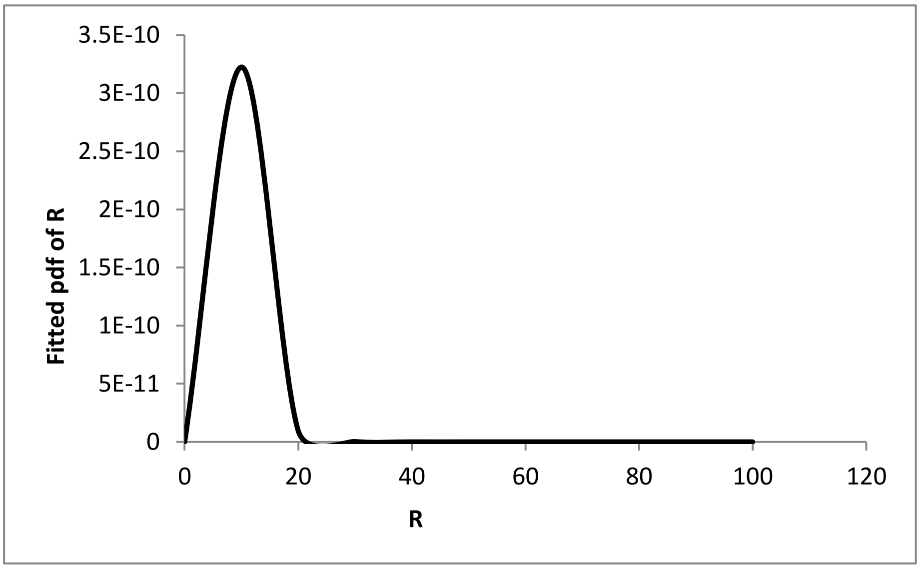


Fig. 16 Fitted pdf of R= $\alpha X/\beta Y$ given by (10) when α=1,β=2 and X is sodium and potassium oxides and Y is iron oxide.

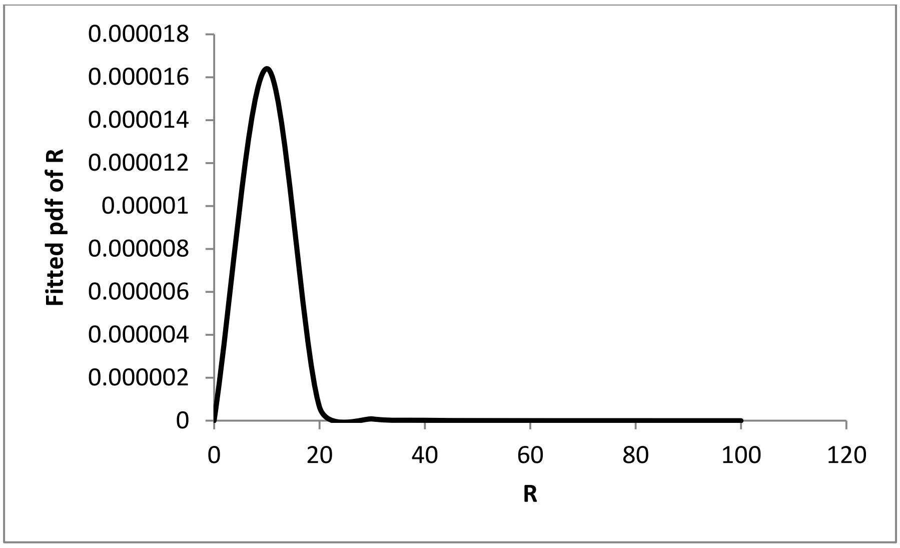


Fig. 17 Fitted pdf of R= *αX/βY* given by (10) when α=1,β=2 and X is sodium and potassium oxides and Y is magnesium oxide.

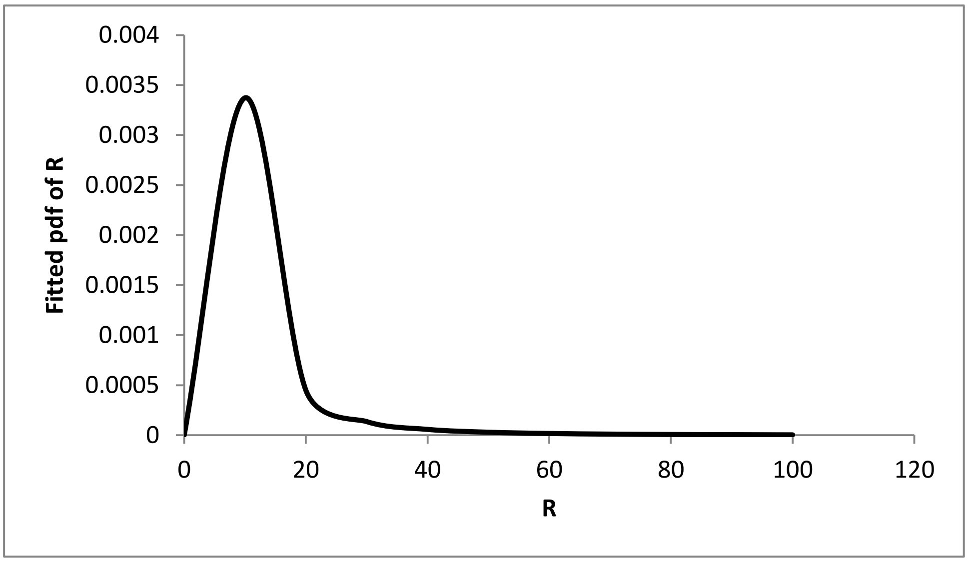


Fig. 18 Fitted pdf of R= *αX/βY* given by (10) when α=1,β=2 and X is iron oxide and Y is magnesium oxide.

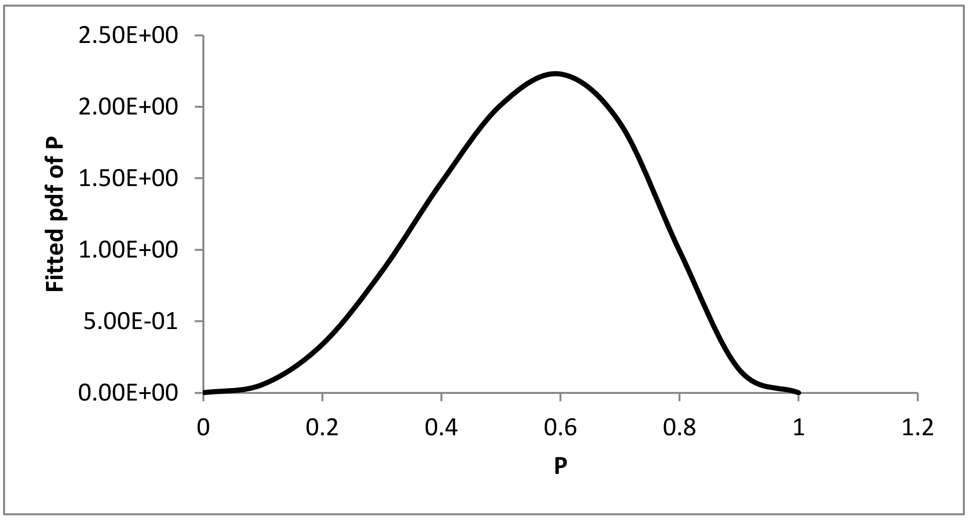


Fig. 19 Fitted pdf of *P* =αβ*XY* given by (12) when α=2,β=2*X* is sodium and potassium oxides and *Y* is iron oxide.

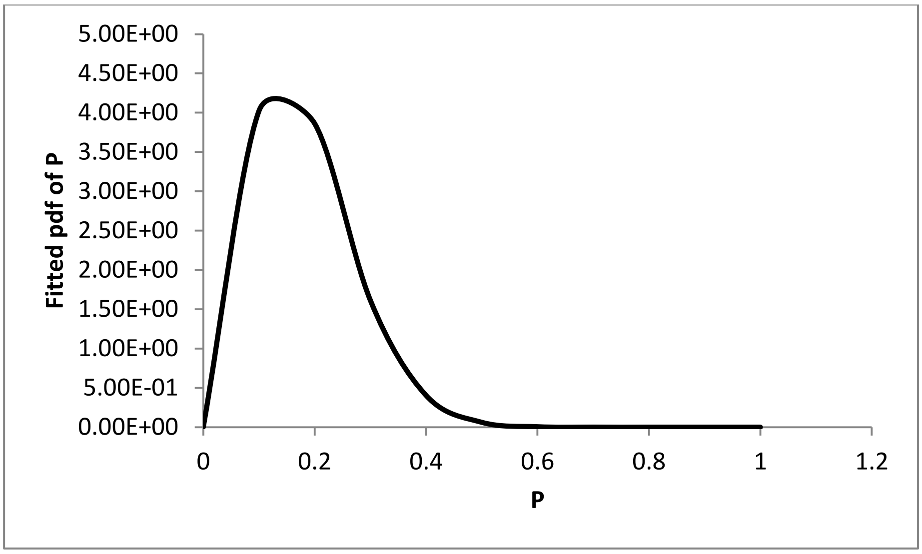


Fig. 20 Fitted pdf of $P = \alpha\beta XY$ given by (12) when $\alpha=2,\beta=2$ X is sodium and potassium oxides and Y is magnesium oxide.

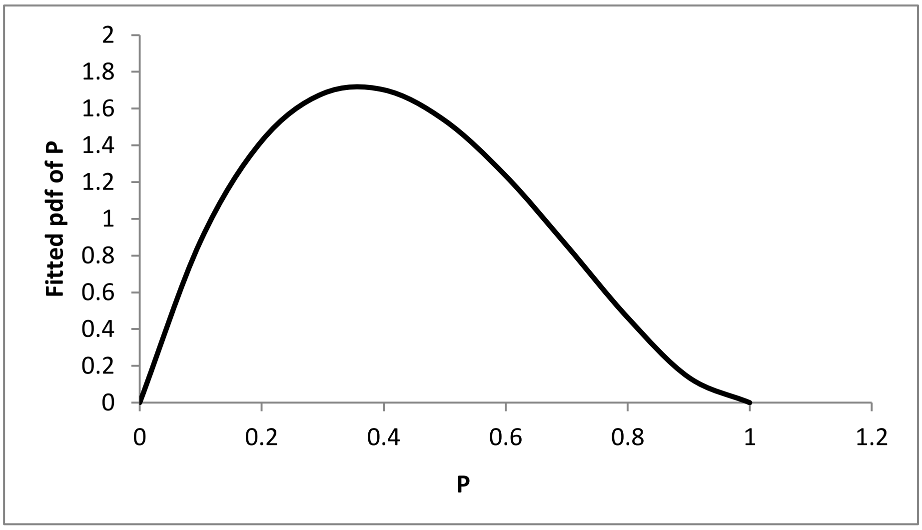


Fig. 21 Fitted pdf of $P = \alpha\beta XY$ given by (12) when $\alpha=2,\beta=2$ X is iron oxide and Y is magnesium oxide.

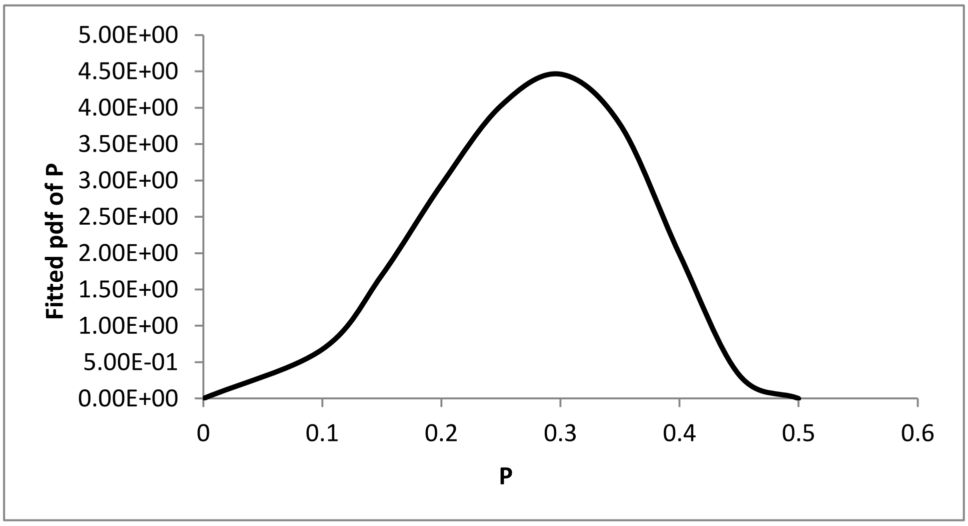


Fig. 22 Fitted pdf of $P = \alpha\beta XY$ given by (12) when $\alpha=2,\beta=1$ X is sodium and potassium oxides and Y is iron oxide.

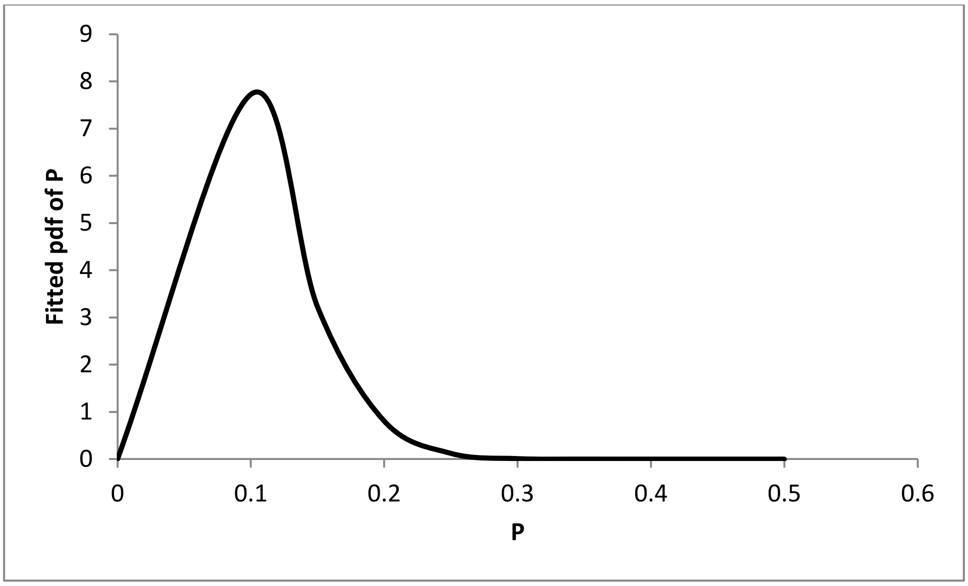


Fig. 23 Fitted pdf of $P = \alpha\beta XY$ given by (12) when $\alpha=2, \beta=1$ X is sodium and potassium oxides and Y is iron oxide.

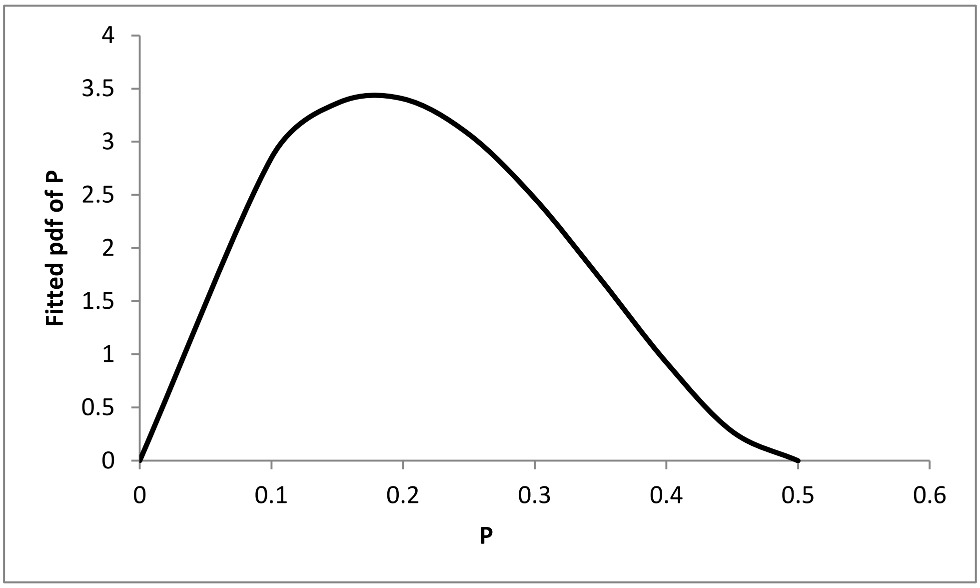


Fig. 24 Fitted pdf of $P = \alpha\beta XY$ given by (12) when $\alpha=2, \beta=1$ X is iron oxide and Y is magnesium oxide.

Clearly from Figures 4-12, 13-18 and 19-24, we can see that the data fits the given distributions.

Conclusions.

The exact distributions of $Z, R \; and \; P$ derived have complex forms involving the Gauss hyper geometric and the Appell function, we deduce approximate distributions for these variables depending on beta type I and type II distributions.

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