جامعة

بنغازي الحديثة

مجلة جامعة بنغازي الحديثة للعلوم والدراسات الإنسانية

مجلة علمية الكترونية محكمة

العــدد التاسع

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شروط كتابة البحث العلمي في مجلة جامعة بنغازي الحديثة للعلوم والدراسات الإنسانية

1- الملخص باللغة العربية وباللغة الانجليزية (150 كلمة).
2- المقدمة، وتشمل التالي:
❖ نبذة عن موضوع الدراسة (مدخل).
❖ مشكلة الدراسة.
❖ أهمية الدراسة.
❖ أهداف الدراسة.
❖ المنهج العلمي المتبع في الدراسة.
3- الخاتمة. (أهم نتائج البحث - التوصيات).
4- قائمة المصادر والمراجع.
5- عدد صفحات البحث لا تزيد عن (25) صفحة متضمنة الملاحق وقائمة المصادر والمراجع.

القواعد العامة لقبول النشر

1. تقبل المجلة نشر البحوث باللغتين العربية والانجليزية؛ والتي تتوافر فيها الشروط الآتية:

- أن يكون البحث أصيلاً، وتتوافر فيه شروط البحث العلمي المعتمد على الأصول العلمية والمنهجية المتعارف عليها من حيث الإحاطة والاستقصاء والإضافة المعرفية (النتائج) والمنهجية والتوثيق وسلامة اللغة ودقة التعبير.
- ألا يكون البحث قد سبق نشره أو قُدم للنشر في أي جهة أخرى أو مستل من رسالة أو اطروحة علمية.
- أن يكون البحث مراعياً لقواعد الضبط ودقة الرسوم والأشكال - إن وجدت - ومطبوعاً على ملف وورد، حجم الخط (14) وبخط (Arial 'Body') للغة العربية. وحجم الخط (12) بخط (Times New Roman) للغة الإنجليزية.
- أن تكون الجداول والأشكال مدرجة في أماكنها الصحيحة، وأن تشمل العناوين والبيانات الإيضاحية.
- أن يكون البحث ملتزما بدقة التوثيق حسب دليل جمعية علم النفس الأمريكية (APA) وتثبيت هوامش البحث في نفس الصفحة والمصادر والمراجع في نهاية البحث على النحو الآتي:
- أن تُثبت المراجع بذكر اسم المؤلف، ثم يوضع تاريخ نشرة بين حاصرتين، ويلي ذلك عنوان المصدر، متبوعاً باسم المحقق أو المترجم، ودار النشر، ومكان النشر، ورقم الجزء، ورقم الصفحة.
- عند استخدام الدوريات (المجلات، المؤتمرات العلمية، الندوات) بوصفها مراجع للبحث: يُذكر اسم صاحب المقالة كاملاً، ثم تاريخ النشر بين حاصرتين، ثم عنوان المقالة، ثم ذكر اسم المجلة، ثم رقم المجلد، ثم رقم العدد، ودار النشر، ومكان النشر، ورقم الصفحة.

2. يقدم الباحث ملخص باللغتين العربية والانجليزية في حدود (150 كلمة) بحيث يتضمن مشكلة الدراسة، والهدف الرئيسي للدراسة، ومنهجية الدراسة، ونتائج الدراسة. ووضع الكلمات الرئيسية في نهاية الملخص (خمس كلمات).

3. تحتفظ مجلة جامعة بنغازي الحديثة بحقها في أسلوب إخراج البحث النهائي عند النشر.

إجراءات النشر

ترسل جميع المواد عبر البريد الالكتروني الخاص بالمجلة جامعة بنغازي الحديثة وهو كالتالي:

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- ✓ يرفق مع البحث نموذج تقديم ورقة بحثية للنشر (موجود على موقع المجلة) وكذلك ارفاق موجز للسيرة الذاتية للباحث إلكترونياً.
- ✓ لا يقبل استلام الورقة العلمية الا بشروط وفورمات مجلة جامعة بنغازي الحديثة.
- ✓ في حالة قبول البحث مبدئياً يتم عرضة على مُحكُمين من ذوي الاختصاص في مجال البحث، ويتم اختيارهم بسرية تامة، ولا يُعرض عليهم اسم الباحث أو بياناته، وذلك لإبداء آرائهم حول مدى أصالة البحث، وقيمته العلمية، ومدى التزام الباحث بالمنهجية المتعارف عليها، ويطلب من المحكم تحديد مدى صلاحية البحث للنشر في المجلة من عدمها.
- ✓ يُخطر الباحث بقرار صلاحية بحثه للنشر من عدمها خلال شهرين من تاريخ الاستلام للبحث، وبموعد النشر، ورقم العدد الذي سينشر فيه البحث.
- ✓ في حالة ورود ملاحظات من المحكُمين، تُرسل تلك الملاحظات إلى الباحث لإجراء التعديلات اللازمة بموجبها، على أن تعاد للمجلة خلال مدة أقصاها عشرة أيام.
- ✓ الأبحاث التي لم تتم الموافقة على نشرها لا تعاد إلى الباحثين.
- ✓ الأفكار الواردة فيما ينشر من دراسات وبحوث وعروض تعبر عن أراء أصحابها.
- ✓ لا يجوز نشر إي من المواد المنشورة في المجلة مرة أخرى.
- ✓ يدفع الراغب في نشر بحثه مبلغ قدره (400 د.ل) دينار ليبي إذا كان الباحث من داخل ليبيا، و (200 $) دولار أمريكي إذا كان الباحث من خارج ليبيا. علماً بأن حسابنا القابل للتحويل هو: **(بنغازي – ليبيا - مصرف التجارة والتنمية، الفرع الرئيسي - بنغازي، رقم 001-225540-0011. الاسم (صلاح الأمين عبدالله محمد)**.
- ✓ جميع المواد المنشورة في المجلة تخضع لقانون حقوق الملكية الفكرية للمجلة.

info.jmbush@bmu.edu.ly

00218913262838

د. صلاح الأمين عبدالله
رئيس تحرير مجلة جامعة بنغازي الحديثة
Dr.salahshalufi@bmu.edu.ly

Pfaffian Differential Equation and It's Solutions

Dr. Saad. A. Salama

(Mathematics Department, Faculty of Science, Tobruk University. Libya)

ABSTRACT

The aim of this paper, is to discuss and solve differential equations with one independent variable and more than one dependent variables, through using given easy method procedure technique to obtain the best solutions, depending on the type of the problem.

Keywords: Pfaffian equation, dependent variable, independent variable, integrability, auxiliary equations, homogeneous equations.

INTRODUCTION

In this paper, we discuss differential equations with one independent variable and more than one dependent variables.

Pfaffian differential equation (denoted by Pf DE). Let $u_i,\ i = 1,2,\ldots,n$ be n functions of some or all of n variables $x_1, x_2, \ldots, x_n$.

Then $\sum_{i=1}^{n} u_i\, dx_i = 0$ is called a Pfaffian differential equation in n variables $x_1, x_2, \ldots, x_n$.

Total (or single)Differential Equations:

An equation of the form $$Pdx + qdy + Rdz = 0 \qquad (1)$$

Where P , Q , R are function of x , y , z is called the single or total differential equation in three variables x , y , z.

Equation (1) cab be directly integrated if there exists a function $u(x, y, z)$ whose total differential du is equal to the left side of (1). In other cases (1) may or may not be integrable. We now proceed to find the condition which P , Q , R must satisfy, so that (1) be integrable. This is called the condition of integrability of the single differential equation (1).

Necessary and sufficient conditions for integrability of the total differential equation:

Let (1) has an integral $$u(x, y, z) = c \qquad (2)$$

Then the total differential du must be equal to Pdx + qdy + Rdz, or to it is multiplied by a factor. But, we know that

$$du = (\partial u/\partial x)dx + (\partial u/\partial y)dy + (\partial u/\partial z)dz. \qquad (3)$$

Since (2) is an integral of (1), P , Q , R should be proportional to $\partial u/\partial x, \partial u/\partial y, \partial u/\partial z$

Therefore $\frac{\partial u/\partial x}{P} = \frac{\partial u/\partial y}{Q} = \frac{\partial u/\partial z}{R} = \lambda(x, y, z)$, since.

$$\lambda P = \partial u/\partial x, \quad \lambda Q = \partial u/\partial y \quad \text{and} \quad \lambda R = \partial u/\partial z. \qquad (4)$$

From the first two equations of (4), we get

$$\frac{\partial}{\partial y}(\lambda P) = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}(\lambda Q)$$

or $$\lambda \frac{\partial P}{\partial y} + P\frac{\partial \lambda}{\partial y} = \lambda \frac{\partial Q}{\partial x} + Q\frac{\partial \lambda}{\partial x} \qquad ie \qquad \lambda\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = Q\frac{\partial \lambda}{\partial x} - P\frac{\partial \lambda}{\partial y} \qquad (5)$$

Similarly $$\lambda\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) = R\frac{\partial \lambda}{\partial y} - Q\frac{\partial \lambda}{\partial z} \qquad (6)$$

and $$\lambda\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) = P\frac{\partial \lambda}{\partial z} - R\frac{\partial \lambda}{\partial x} \qquad (7)$$

By multiplying (5), (6) and (7) by R, P and Q respectively and adding to each other, it follows that

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0 \qquad (8)$$

This is, the necessary and sufficient condition for integrability of the equation(1).

Sufficient condition: Suppose that the coefficients P , Q , R satisfy the relation (8). Now it can be proved that this relation gives the required sufficient condition for the existence of an integral of (1). For this condition it can be shown that an integral of (1) can be found when relation (8) holds.

We first prove that if we take $P_1 = \mu P, Q_1 = \mu Q, R_1 = \mu R$, where μ is any function of x , y and z, the same condition is satisfied by both groups P_1 , Q_1 , R_1 and P , Q , R we have

$$\frac{\partial Q_1}{\partial z} - \frac{\partial R_1}{\partial y} = \mu\frac{\partial Q}{\partial z} + Q\frac{\partial \mu}{\partial z} - \left(\mu\frac{\partial R}{\partial y} + R\frac{\partial \mu}{\partial y}\right), \text{ as } Q_1 = \mu Q \text{ and } R_1 = \mu R$$

or $$\frac{\partial Q_1}{\partial z} - \frac{\partial R_1}{\partial y} = \mu\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\frac{\partial \mu}{\partial z} - R\frac{\partial \mu}{\partial y} \qquad (9)$$

Similarly $$\frac{\partial R_1}{\partial x} - \frac{\partial P_1}{\partial z} = \mu\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\frac{\partial \mu}{\partial x} - P\frac{\partial \mu}{\partial z} \qquad (10)$$

and $$\frac{\partial P_1}{\partial y} - \frac{\partial Q_1}{\partial x} = \mu\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) + P\frac{\partial \mu}{\partial y} - Q\frac{\partial \mu}{\partial x} \qquad (11)$$

Multiplying (9), (10) and (11) by P_1, Q_1, R_1 respectively, adding, and replacing P_1, Q_1, R_1 by $\mu P, \mu Q, \mu R$ respectively in resulting R.H.S., we obtain

$$P_1\left(\frac{\partial Q_1}{\partial z} - \frac{\partial R_1}{\partial y}\right) + Q_1\left(\frac{\partial R_1}{\partial x} - \frac{\partial P_1}{\partial z}\right) + R_1\left(\frac{\partial P_1}{\partial y} - \frac{\partial Q_1}{\partial x}\right)$$

$$= \mu\left\{P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)\right\} = 0 \qquad (12)$$

Now $Pdx + Qdy$ may be regarded as an exact differential. For if is not so, then Multiplying the equation (1) by the integrating factor (9), (10) and (11) by $\mu(x, y, z)$, we can make it so. Thus there is not loss of generality in regarding $Pdx + Qdy$ as an exact differential. For this the condition is:

$$\partial P/\partial y = \partial Q/\partial x \tag{13}$$

Let $$V = \int (Pdx + Qdy) \tag{14}$$

then it follows that $$P = \partial V/\partial x \quad and \quad Q = \partial V/\partial y \tag{15}$$

From (15) $\frac{\partial P}{\partial z} = \frac{\partial^2 V}{\partial z \partial x}$ and $\frac{\partial Q}{\partial z} = \frac{\partial^2 V}{\partial z \partial y}$

Using the above relation, (13) and (15), (8) gives

$$\frac{\partial V}{\partial x}\left(\frac{\partial^2 V}{\partial z \partial x} - \frac{\partial R}{\partial y}\right) + \frac{\partial V}{\partial y}\left(\frac{\partial R}{\partial x} - \frac{\partial^2 V}{\partial z \partial x}\right) = 0 \quad \text{or} \quad \frac{\partial V}{\partial x}\frac{\partial}{\partial y}\left(\frac{\partial V}{\partial z} - R\right) - \frac{\partial V}{\partial y}\frac{\partial}{\partial x}\left(\frac{\partial V}{\partial z} - R\right) = 0$$

or
$$\begin{vmatrix} \frac{\partial V}{\partial x} & \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial z} - R\right) \\ \frac{\partial V}{\partial y} & \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial z} - R\right) \end{vmatrix} = 0$$

This shows that a relation independent of x and y exists between V ands $(\partial V/\partial z) - R$.

Consequently $(\partial V/\partial z) - R$ can be expressed as a function of z and V alone. That is, we can take $(\partial V/\partial z) - R = \emptyset(z, V)$ (16)

Now. $Pdx + qdy + Rdz = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \left(\frac{\partial V}{\partial z} - \emptyset\right)dz,$ using (14) and (16)

$$= \left(\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz\right) - \emptyset dz = dV - \emptyset dz.$$

Thus (1) may be written as $dV - \emptyset dz = 0$ which is an equation in two variable. Hence its integration will give an integral of the form

$$F(V, z) = 0,$$

Hence the condition (8) is sufficient.

Thus (8) is both the necessary and sufficient condition that (1) has an integral.

Theorem: Prove that the necessary and sufficient conditions for integrability of the total differential equation

$$A.dr = Pdx + Qdy + Rdz = 0 \text{ is } A.curl\, A = 0$$

Proof: Given $$A.dr = Pdx + Qdy + Rdz = 0 \tag{17}$$

Let $$r = xi + yj + zk \quad so\ that \quad dr = dxi + dyj + dzk \tag{18}$$

and $$A = Pi + Qj + Rk \tag{19}$$

Then we see that (17) satisfied by usual rule of dot product of two vectors A and dr.

Now show that the necessary condition for integrability of (17) is

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0 \tag{20}$$

From vector calculus, we know that

Benghazi Modern University
info.jmbush@bmu.edu.ly Dr.salahshalufi@bmu.edu.ly
Mobile +218945429096

$$Curl\ A = \left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) i + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) j + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) k \qquad (21)$$

Hence, using (19) , (21) and applying the usual rule of dot product of two vectors, the necessary condition (20) may be rewritten as,

$A.\,curl\ A = 0$ as desired.

The conditions for exactness of (1):

The given total differential equation is said to be exact if the following three conditions are satisfied

$$\partial P/\partial y = \partial Q/\partial x,\ \ \partial Q/\partial z = \partial R/\partial y \ \text{ and } \ \partial R/\partial x = \partial P/\partial z. \qquad (22)$$

Note that when conditions (22) are satisfied, the conditions for integrability of equation (1) is also satisfied, for each term of (20), vanishes identically.

Methods of solving equation (1):

There are several methods of solving (1). We know that (1) integrable when the following conditions satisfied equation(20).

1) Special Method I Solution homogeneous equation:

The equation (1) is called homogeneous equation if P, Q, R are homogeneous functions of x, y, z of the same degree.

There are two following working rules to solve such equations.

Working rule I. Solution by use an integrating factor (I.F.):

Step 1: As usual, verify that the given is integrable.

Step 2: Calculate $Pdx + Qdy + Rdz$. If it is not equal to zero, then $1/(Pdx + Qdy + Rdz)$ is taken as I.F. (Integrating factor) of the given equation. I.F.=$1/D$, where $D = Pdx + Qdy + Rdz$.

Step 3: Multiply the given equation by I.F. (1/D) where D denotes denominator of I.F.

Find $d(D) i.e.$ total differential of D. Now add and subtract $d(D)$ from the numerator.

Write the given equation in the form $\frac{d(D)\pm\cdots}{D} = 0$ or $\frac{d(D)}{D} \pm \cdots = 0$ and then integrate.

Working rule II: The first method fails when equation (1), in such cases we apply the following method which is applicable to all homogeneous equations.

Step 1: Do same as done in step 1 of working rule 1.

Step 2: Put $x = zu,\ \ y = zv$ so that $dx = udz + zdu$ and $dy = zdv + vdz$.

Substituting these in the given equation two cases may arise.

Case I. if the coefficient of dz is zero, we shall have an equation in only two variables u and v. By regrouping properly, it can be easily integrated.

Case II. if the coefficient of dz is not zero, then we shall be able to separate z from u and v. Thus the resulting equation will be of the form

$$\frac{f_1(u,v)du+f_2(u,v)dv}{f(u,v)}+\frac{dz}{z}=0. \quad \text{(a)}$$

We now denote $f(u,v)$ by D and find $d(D)$. Add and subtract $d(D)$ as done in step 3 of the working rule 1. Finally we integrate, after integration u and v are replaced by x/z and y/z respectively so as to get the desired solution in x, y and z.

Note. Sometimes integration of (a) is possible without assuming D etc. Hence we use D only when it helps to integrate equation (a).

Examples:

Example(1): *Solve* $(yz+z^2)dx-xzdy+xydz=0$

Sol. Given $(yz+z^2)dx-xzdy+xydz=0$ (i)

Comparing the given equation with $Pdx+Qdy+Rdz=0$, we have

$P=yz+z^2$, $Q=-xz$, $R=xy$ and let $D=Pdx+Qdy+Rdz$.

The condition of intgrability is satisfied

Now, $D=x(yz+z^2)-xyz+xyz=xz(y+z)\neq 0$ (ii)

Multiplying (i) by integrating factor $1/D$, $\frac{(yz+z^2)dx-xzdy+xydz}{D}=0$ (iii)

Now $d(D)=d[xz(y+z)]=(zdx+xdz)(y+z)+xz(dy+dz)$

or $d(D)=z(y+z)dx+x(y+2z)dz+xzdy.$ (iv)

Re-writing, the numerator of (iii)

$=d(D)-d(D)+(yz+z^2)dx-xzdy+xydz=d(D)-2xz(dy+dz)$, by (d).

$\therefore$ (iv) becomes $\frac{d(D)}{D}-\frac{2xz(dy+dz)}{D}=0$ or $\frac{d(D)}{D}-\frac{2xz(dy+dz)}{xz(y+z)}=0$

so that $\frac{d(D)}{D}-\frac{2(dy+dz)}{y+z}=0.$

Integrating, $logD-2\,log(y+z)=logc$ or $D=c(y+z)^2$

or $xz(y+z)=c(y+z)^2$ or $xz=c(y+z)$,

which is the required solution, c being an arbitrary constant.

Verify the usual condition of integrability.

Let $x=uz$ and $y=vz$ so that $dx=udz+zdu$ and $dy=vdz+zdv$

Putting these values of x, y, dx, dy in (i), we get

$$(vz^2+z^2)(udz+zdu)-uz^2(vdz+zdv)+vuz^2dz=0$$

$$(v+1)z^3du-uz^3dv+(v+1)uz^2dz=0.$$

Dividing by $u(v+1)z^3$ we get $\frac{du}{u} - \frac{dv}{1+v} + \frac{dz}{z} = 0$

Integrating, $log(u) = log(v+1) + log(z) = log(c)$ or $uz = c(v+1)$

or $(x/z)z = c(1+y/z)$ or $xz = c(y+z)$ as $u = x/z$ $and v = y/z$

Example(2):

$$Solve (2xz - yz)dx + (2yz - xz)dy - (x^2 - xy + y^2)dz = 0$$

Sol: As usual, verify that the given equation is integrable.

Since the given equation is homogeneous, we put

$x = uz$ and $y = vz$ so that $dx = udz + zdu$ and $dy = vdz + zdv$ (i)

Putting these in the given equation, we get

$$(2uv^2 - vz^2)(udz + zdu) + (2vz^2 - uz^2)(vdz + zdv)$$
$$-(u^2z^2 - uvz^2 + v^2z^2)dz = 0$$

or $(2u - v)(udz + zdu) + (2v - u)(vdz + zdv) - (u^2 - uv + v^2)dz = 0$

or $z[(2u - v)du + (2v - u)dv]$
$$+[u(2u - v) + v(2v - u) - (u^2 - uv + v^2)]dz = 0$$

or $z[2udu - (udv + vdu) + 2vdv] + (u^2 - uv + v^2)dz = 0$

or $z[du^2 - d(uv) + dv^2] + (u^2 - uv + v^2)dz = 0$

or $\frac{d(u^2 - uv + v^2)}{u^2 - uv + v^2} + \frac{dz}{z} = 0$

Integrating, $log((u^2 - uv + v^2) + log(z) = log(c)$

or $z((u^2 - uv + v^2) = c$ or $z\left(\frac{x^2}{z^2} - \frac{x}{z} \times \frac{y}{z} + \frac{y^2}{z^2}\right) = c$ or $x^2 - xy + y^2 = cz$

2) Special Method II use of auxiliary equations:

Computing (1) and (8), we obtain simultaneous equation, known as auxiliary equations

$$\frac{dx}{\partial Q/\partial z - \partial R/\partial y} = \frac{dy}{\partial R/\partial x - \partial P/\partial z} = \frac{dz}{\partial P/\partial y - \partial Q/\partial x} \qquad (a)$$

Let $u = c_1$ and $v = c_2$ be two integrals, the following equation

$$Adu + Bdv = 0 \qquad (b)$$

Compare (1) and (b) and thus get values of A and B. Put these values of A and B in (b) and then integrate the resulting equation. Now substitute the values of u and v in the relation after integration. We thus obtain the required general solution.

Note 1: Method II discussed will fail in case the equation (1) in exact, $i.e.$, when

$$\partial Q/\partial z = \partial R/\partial y \ , \ \partial R/\partial x = \partial P/\partial z \ \ and \ \ \partial P/\partial y = \partial Q/\partial x$$

Note 2: This method is generally applied when solution by method 1is not convenient.

Example: Solve

$$(y^2 + yz + z^2)dx + (z^2 + zx + x^2)dy + (x^2 + xy + y^2)dz = 0$$

Sol. Given

$$(y^2 + yz + z^2)dx + (z^2 + zx + x^2)dy + (x^2 + xy + y^2)dz = 0 \qquad \text{(i)}$$

Comparing (i) with $Pdx + Qdy + Rdz = 0$, we have

$$P = y^2 + yz + z^2, \quad Q = z^2 + zx + x^2, \quad R = x^2 + xy + y^2 \qquad \text{(ii)}$$

The auxiliary equations of the given equation are

$$\frac{dx}{\partial Q/\partial z - \partial R/\partial y} = \frac{dy}{\partial R/\partial x - \partial P/\partial z} = \frac{dz}{\partial P/\partial y - \partial Q/\partial x}$$

$$\frac{dx}{(x+2z)-(x+2y)} = \frac{dy}{(2x+y)-(y+2z)} = \frac{dz}{(2y+z)-(2x+z)}$$

$$\frac{dx}{z-y} = \frac{dy}{x-z} = \frac{dz}{y-x} \qquad \text{(iii)}$$

Each member of (iii) $= \frac{dx+dy+dz}{z-y+x-z+y-x} = \frac{dx+dy+dz}{0}$

so that $dx + dy + dz = 0$

Integrating, $\quad x + y + z = c_1 = u \qquad$ (iv)

Again, using multipliers $z + y, x + z, y + x$, each member of (iii)

$$= \frac{(z+y)dx+(x+z)dy+(y+x)dz}{(z+y)(z-y)+(x+z)(x-z)+(y+x)(y-x)} = \frac{(xdy+ydx)+(ydz+zdy)+(zdx+xdz)}{0} = \frac{d(xy)+d(yz)+d(zx)}{0}$$

so that $d(xy) + d(yz) + d(zx) = 0$ integrating $xy + yz + zx = c_2 = v \qquad$ (v)

We now proceed to determine two functions A and B in such a manner so that given equation (i) becomes identical with $Adu + Bdv = 0 \qquad$ (vi)

Using (iv) and (v), (vi) reduces to $Ad(x + y + z) + Bd(xy + yz + zx) = 0$

or $\quad A(dx + dy + dz) + B(ydx + xdy + zdy + ydz + xdz + zdx) = 0$

or $\quad \{A + B(y + z)\}dx + \{A + B(z + x)\}dy + \{A + B(x + y)\}dz = 0 \qquad$ (vii)

Comparing (vii) with (i), we have

$$A + B(y + z) = y^2 + yz + z^2 \qquad \text{(viii)}$$

$$A + B(z + x) = z^2 + xz + x^2 \qquad \text{(ix)}$$

$$A + B(x + y) = x^2 + xy + y^2 \qquad \text{(x)}$$

Subtracting (ix) from (viii), we get

$$\mathrm{B(y - x) = y^2 - x^2 + z(y - x) = (y - x)(x + y + z)}$$

$\therefore \qquad B = x + y + z = u$ by (iv) $\qquad$ (xi)

From (viii) $\qquad A = y^2 + yz + z^2 - B(y + z)$

$\qquad = y^2 + yz + z^2 - (x + y + z)(y + z)$ by (i)

$\qquad = y^2 + yz + z^2 - (y^2 + z^2 + 2yz + xy + zx)$

Thus, $A = -(xy + yz + zx)$ or $A = -v$ by (5) (xii)

Using (xi) and (xii), (vi) becomes

$$-vdu + udv = 0 \quad \text{or} \quad (1/v)dv = (1/u)du$$

Integrating $log(v) = log(u) + log(c)$ or $v = cu$

or $xy + yz + zx = c(x + y + z)$ by (iv) and (v).

3) General Method III of solving equation (1) by taking one variable as constant:

Step 1: First verify the condition of integrability.

Step 2: we now treat one of the variables, say z, as a constant i. e. $\mathrm{dz} = 0$, then the resulting equations is reduced to $pdx + qdy = 0$. (a)

We should select a proper variable to be constant so that the resulting equation in the remaining variables is easily integrable. Thus this selection will vary from problem to problem. The present discussion is for the choice $\mathrm{z} = \mathrm{constsnt}$. For other cases the necessary changes have to be made in the entire procedure.

Step 3: Let the solution of (a) by $u(x, y) = f(z)$, where $f(z)$ is an arbitrary function of z. Note that in place of taking merely an absolute constant, we have taken $f(z)$.This is possible because the arbitrary function $f(z)$ is constant with respect to x and y. This is in keeping without starting assumption, namely $z = constant$. Thus the solution of (a) is of the form

$$u(x, y) = f(z) \quad \text{(b)}$$

Step 4: We now differentiate (b) totally with respect to x, y, z and then compare the result with equation (1).

After comparing we shall get an equation in two variables f and z. If the coefficient of df or dz involve functions of x and y, it will always be possible to remove them by using (b).

Step 5: Solve the equation got in step 4 and obtain f. Putting this value of f in (b), we shall get the required solution of the given equation.

Remarks: Many equations can be solved by this method, but the method may be become tedious in some problems.

We shall apply this method whenever there is no difficulty in solving the equation obtained by treating one variable as constant.

Examples:

Example(1): Solve $y^2z(xcosx - sinx)dx + x^2z(ycosy - siny)dy$

$$+xy(ysinx + xsiny + xycosz)dz = 0$$

Sol. As usual verify that the given equation is integrable. Let z be treated as constant, so that $dz = 0$. Then the given equation becomes

$$y^2z(xcosx - sinx)dx + x^2z(ycosy - siny)dy = 0$$

or $\frac{xcosx-sinx}{x^2}dx+\frac{ycosy-siny}{y^2}dy=0,\quad d\left(\frac{sinx}{x}\right)+d\left(\frac{siny}{y}\right)=0$

Integrating it, $\frac{sinx}{x}+\frac{siny}{y}=f(z)$ say (i)

where f(z) is taken as constant of integration as z is treated as constant.

Differentiating (i), $\frac{xcosx-sinx}{x^2}dx+\frac{ycosy-siny}{y^2}dy=f'(z)dz$

or $zy^2(xcosx-sinx)dx+zx^2(ycosy-siny)dy-x^2y^2zf'(z)=0$ (ii)

Comparing (ii) with the given equation, we have

$$-x^2y^2zf'(z)=xy(ysinx+xsiny+xycosz)$$

or $-zf'(z)=\frac{sinx}{x}+\frac{siny}{y}+cosz=f(z)+cosz$, using (i)

or $\frac{df}{dz}+\frac{1}{z}f=-\frac{cosz}{z}$ which is a linear differential equation.

Its I. F. $=e^{\int(1/z)dz}=e^{log(z)}=z$ and its solution is

$zf(z)=\int z\left(-\frac{cosz}{z}\right)dz+c=-sinz+c$ or $z\left(\frac{sinx}{x}+\frac{siny}{y}\right)=c-sinz$ using (i).

Example(2): Verify that the following equation is integrable and find its primitive

$$zydx+(x^2y-zx)dy+(x^2z-xy)dz=0$$

Sol. Given $zydx+(x^2y-zx)dy+(x^2z-xy)dz=0$ (i)

Treating x as constant so that $dx=0$, (i) reduces to

$$(x^2y-zx)dy+(x^2z-xy)dz=0$$

or $x^2(ydy+zdz)-x(zdy+ydz)=0$ (ii)

integrating (ii) and remembering that x is being regarded as constant, we get $(x^2/2)(y^2+z^2)-xyz=f(x)$, f being an arbitrary function. (iii)

differentiating (iii), we have

$$xdx(y^2+z^2)+(x^2/2)(2ydy+2zdz)-xydz-yzdx-zxdy=f'(x)dx$$

or $[x(y^2+z^2)-yz-f'(x)]dx+(x^2y-xy)dy+(x^2z-xy)dz=0$

comparing the above equation with (i), we have

$x(y^2+z^2)-yz-f'(x)=yz$ or $x(y^2+z^2)-2yz=f'(x)$

or $(x^2/2)(y^2+z^2)-xyz=(x/2)\times f'(x)$ or $f=(x/2)\times f'(x)$ by (iii)

or $f=(x/2)\times(df/dx)$ or $(1/f)df=(2/x)dx$ so that $f(x)=cx^2$

Putting this value of $f(x)$ in (iii), the required primitive is

$(x^2/2)(y^2+z^2)-xyz=cx^2$ or $x^2(y^2+z^2-2c)=2xyz.$

4) Solution of (1) if it is exact and homogeneous of degree n $\neq$ 1:

Theorem: $xP+yQ+zR=c$ is the solution of equation (1), when it is exact and homogeneous of degree $n\neq 1$.

Proof: Given solution is $xP+yQ+zR=c$ (a)

Differentiating (a), we obtain

$$\left(P + x\frac{\partial P}{\partial x} + y\frac{\partial Q}{\partial x} + z\frac{\partial R}{\partial x}\right)dx + \left(x\frac{\partial P}{\partial y} + Q + y\frac{\partial Q}{\partial y} + z\frac{\partial R}{\partial y}\right)\mathrm{dy}$$

$$+\left(x\frac{\partial P}{\partial z} + y\frac{\partial Q}{\partial z} + R + z\frac{\partial R}{\partial z}\right)dz = 0 \qquad \text{(b)}$$

Since equation (1) is exact, we have

$$\partial P/\partial y = \partial Q/\partial x\,,\ \partial Q/\partial z = \partial R/\partial y \ \text{ and } \ \partial R/\partial x = \partial P/\partial z \qquad \text{(c)}$$

Using relation (c) , (b) may be re-written as

$$\left(P + x\frac{\partial P}{\partial x} + y\frac{\partial P}{\partial y} + z\frac{\partial P}{\partial z}\right)dx + \left(Q + x\frac{\partial Q}{\partial x} + y\frac{\partial Q}{\partial y} + z\frac{\partial Q}{\partial z}\right)dy$$

$$+\left(R + x\frac{\partial R}{\partial x} + y\frac{\partial R}{\partial y} + z\frac{\partial R}{\partial z}\right)dz = 0 \qquad \text{(d)}$$

Since equation (1) is homogeneous of degree n, it follows that P , Q and R are all homogeneous functions of degree n. Using Euler's theorem on homogeneous functions $P(x,y,z)$, $Q(x,y,z)$ and $R(x,y,z)$ of degree n we get

$$\left.\begin{aligned} x(\partial P/\partial x) + y(\partial P/\partial y) + z(\partial P/\partial z) &= nP \\ x(\partial Q/\partial x) + y(\partial Q/\partial y) + z(\partial Q/\partial z) &= nQ \\ x(\partial R/\partial x) + y(\partial R/\partial y) + z(\partial R/\partial z) &= nR \end{aligned}\right\} \qquad \text{(e)}$$

and

Using (e), (d) reduces to

$$(P + nP)dx + (Q + nQ)dy + (R + nR)dz = 0$$

or $\quad (n + 1)(Pdx + Qdy + Rdz) = 0$

or $\quad (Pdx + Qdy + Rdz) = 0,\ $ as $n \neq -1$ so that $(n + 1) \neq 0$ (f)

which is given differential equation and hence (a) is solution of (f) as required.

Example : Solve $(x - 3y - z)dx + (2y - 3x)dy + (z - x)dz = 0$

Sol. Given $(x - 3y - z)dx + (2y - 3x)dy + (z - x)dz = 0$ (i)

Comparing (i) with $Pdx + Qdy + Rdz = 0$, we have

$$P = x - 3y - z, \qquad Q = 2y - 3x \qquad \text{and} \qquad R = z - x \qquad \text{(ii)}$$

(i) is homogeneous equation of degree $n = 1 \neq -1$. Also from (ii), we get

$$(\partial P/\partial y) = -3 = \partial Q/\partial x \quad , \quad (\partial Q/\partial z) = 0 = \partial R/\partial y,$$

$$(\partial R/\partial x) = -1 = \partial P/\partial z \qquad \text{(iii)}$$

(iii) shows that (i) is exact. Thus, (i) is exact and homogeneous of degree $n = 1 \neq -1$. Hence, solution of (i) is give by $xP + yQ + zR = c$

or $\quad x(x - 3y - z) + y(2y - 3x) + z(z - x) = c$

or $\quad x^2 + 2y^2 + z^2 - 6xy - 2xz = c,\ $ c being an arbitrary constant.

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